Final Exam

MAT 552

December 2004

Name:	**ID #:**

This final exam summarizes all the material we have studied so far. It is a take-home exam; you should hand in the solutions by 5 pm Th, Dec. 16. I'll be in my office; if I am not, slip your solution under the door.

These problems should be solved using only the results discussed in class; you are allowed to check your notes or books to review these results. However, please do not try to dig in the books for more general results you could use — this would make some problems trivial.

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1. Let M be the set of all $n \times n$ marices of rank 1. Show that M has a natural structure of a smooth manifold; calculate its dimension and the tangent space $T_A M$, where $A = E_{11} \in M$. (Hint: M is a homogeneous space.)

2. Let G be the group of all affine transformations of $\mathbb{R}$, i.e. all maps $f : \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$, $a \neq 0$. Describe explicitly the corresponding Lie algebra $\mathfrak{g}$ and the exponential map. Is $\mathfrak{g}$ semisimple? solvable? nilpotent?

3. Let V be an n-dimensional complex vector space and B — a symmetric bilinear form on V of rank $r < n$. Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be the group of linear transformations preserving B.
   (a) Describe the corresponding Lie algebra and find its dimension.
   (b) Write decomposition $\mathfrak{g} \simeq \mathfrak{g}_{ss} \oplus \mathfrak{b}$ (direct sum as vector spaces), where $\mathfrak{g}_{ss}$ is a semisimple subalgebra in $\mathfrak{g}$ and $\mathfrak{b}$ is a solvable ideal.

4. Let G be a compact real Lie group, $\mathfrak{g}$ — the corresponding Lie algebra, and V — a complex finite-dimensional representation. Show that for every $x \in \mathfrak{g}$, its action in V is diagonalizable. Is the same true for any $x \in \mathfrak{g}_{\mathbb{C}}$?

5. Show that the vector space
$$V = S^k\mathbb{C}^n = \{\text{homogeneous polynomials in } x_1, \dots, x_n \text{ of degree } k\}$$
has a natural structure of $\mathfrak{sl}(n, \mathbb{C})$-module. Show that it is irreducible and find the highest weight (hint: find all vectors $v \in V$ such that $\mathfrak{n}_+ v = 0$). Find dimension of the zero weight space $V[0]$.

6. Let $\mathfrak{g}$ be a semisimple complex Lie algebra, $(\, , \,)$ — the Killing form, and
$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$
the root decomposition. For any $\alpha \in R_+$, let $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ be such that $(e_\alpha, f_\alpha) = 1$, and let x_i be an orthonormal basis in $\mathfrak{h}$.
   (a) Show that
$$C = \sum_{\alpha \in R+} (e_\alpha f_\alpha + f_\alpha e_\alpha) + \sum x_i^2 \in U\mathfrak{g}$$
is central in $U\mathfrak{g}$.
   (b) Calculate the value of C in the irreducible highest-weight representation L_λ.