MAT 552: PROBLEM SET 4
DUE TUESDAY 10/26

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Unless otherwise specified, the word "representation" means a finite-dimensional complex representation.

1. Let $C = ef + fe + \frac{1}{2}h^2 \in U\ \mathfrak{sl}(2, \mathbb{C})$.

(a) Show that C is central.

(b) Find the eigenvalues of C in each of the representations S^kV defined in the previous homework.

(c) Recall that we have an isomorphism $\mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})$ which gives isomorphism of the corresponding enveloping algrebras. Show that this isomorphism identifies element C above with a multiple of $J_x^2 + J_y^2 + J_z^2$.

2. Let $\mathfrak{g}$ be a complex Lie algebra with a non-degenerate invariant symmetric bilinear form B, and let x_i be an orthonormal basis in $\mathfrak{g}$ with respect to this form. Show that the *Casimir element* $C = \sum x_i^2 \in U\mathfrak{g}$ is central. (Hint: in the basis x_i, every $\operatorname{ad} y$ is given by a skew-symmetric matrix).

3. (a) Show that $(x, y) = \operatorname{tr}(xy)$ is a negative definite symmetric bilinear form on $\mathfrak{su}(n)$ which is invariant under adjoint action of $\mathrm{SU}(n)$.

(b) Let $\mathfrak{g}$ be a real Lie algebra which admits a symmetric positive-definite invariant bilinear form. Deduce from this that the Killing form $(x, y) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} x \operatorname{ad} y)$ is negative semidefinite; if $\mathfrak{g}$ has no center, then it is negative definite.

4. Let G be a connected compact Lie group with discrete center. Show that then there exists an $\operatorname{Ad} G$-invariant symmetric bilinear positive definite form on $\mathfrak{g}$, and deduce from it that the Killing form on $\mathfrak{g}$ is negative definite.

5. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ be the subspace consisting of block-triangular matrices:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right\}$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix, and D is a $(n-k) \times (n-k)$ matrix.

(a) Show that $\mathfrak{g}$ is a Lie subalgebra (this is a special case of so-called *parabolic subalgebras*).

(b) Show that radical $\mathfrak{r}$ of $\mathfrak{g}$ consists of matrices of the form $\begin{pmatrix} \lambda \cdot I & B \\ 0 & \mu \cdot I \end{pmatrix}$, and describe $\mathfrak{g}/\mathfrak{r}$.