MAT 552: PROBLEM SET 3
DUE TUESDAY 10/12

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Unless otherwise specified, the word "representation" means a finite-dimensional complex representation.

1. Let $V = \mathbb{C}^2$ be the standard 2-dimensional representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, and let S^kV be the symmetric power of V.

(a) Write explicitly the action of $e, f, h \in \mathfrak{sl}(2,\mathbb{C})$ (see notation of the previous homework) in the basis $e_1^i e_2^{k-i}$.

(b) Show that S^2V is isomorphic to the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$.

(c) By results of the previous homework, each represenation of $\mathfrak{sl}(2,\mathbb{C})$ can be considered as a representation of $\mathfrak{so}(3,\mathbb{R})$. Which of representations S^kV can be lifted to a representation of $\mathrm{SO}(3,\mathbb{R})$?

2. Let V be an irreducible representation of a Lie algebra $\mathfrak{g}$. Show that the space of $\mathfrak{g}$-invariant bilinear forms on V is either zero or 1-dimensional.

3. Let $\mathfrak{g}$ be a Lie algebra, and $(\, , \,)$ — a symmetric ad-invariant bilinear form on $\mathfrak{g}$. Show that the element $\omega \in (\mathfrak{g}^*)^{\otimes 3}$ given by

$$\omega(x, y, z) = ([x, y], z)$$

is skew-symmetric and ad-invariant.

4. Prove that if $A\colon \mathbb{C}^n \to \mathbb{C}^n$ is an operator of finite order: $A^k = I$ for some k, then A is diagonalizable. [Hint: use theorem about complete reducibility of representations of a finite group]

5. Let C be the standard cube in $\mathbb{R}^3$: $C = \{|x_i| \le 1\}$, and let S be the set of faces of C (thus, S consists of 6 elements). Consider the 6-dimensional complex vector V space of functions on S, and define $A\colon V \to V$ by

$$(Af)(\sigma) = \frac{1}{4}\sum_{\sigma'} f(\sigma')$$

where the sum is taken over all faces σ' which are neighbors of σ (i.e., have a common edge with σ). The goal of this problem is to diagonalize A.

(a) Let $G = \{g \in \mathrm{O}(3,\mathbb{R}) \mid g(C) = C\}$ be the group of symmetries of C. Show that A commutes with the natural action of G on V.

(b) Let $z = -I \in G$. Show that as a representation of G, V can be decomposed in the direct sum

$$V = V_+ \oplus V_-, \qquad V_\pm = \{f \in V \mid zf = \pm f\}$$

(c) Show that as a representation of G, V_+ can be decomposed in the direct sum

$$V_+ = V_+^0 \oplus V_+^1, \quad V_+^0 = \{f \in V_+ \mid \sum_\sigma f(\sigma) = 0\}, \quad V_+^1 = \mathbb{C}\cdot 1$$

where 1 denotes the constant function on S whose value at every $\sigma \in S$ is 1.

(d) Find the eigenvalues of A on V_-, V_+^0, V_+^1.

[Note: in fact, each of V_-, V_+^0, V_+^1 is an irreducible representation of G, but you do not need this fact.]

6. Let $G = \mathrm{SU}(2)$. Recall that we have a diffeomorphism $G \simeq S^3 \subset \mathbb{R}^4$.

(a) Show that the left action of G on $G = S^3$ can be extended to an action of G by orthogonal transformations on $\mathbb{R}^4$.

(b) Let $\omega \in \Omega^3(G)$ be a left-invariant 3-form whose value at $1 \in G$ is defined by

$$\omega(x_1, x_2, x_3) = \mathrm{tr}([x_1, x_2]x_3), \qquad x_i \in \mathfrak{g} = T_1G$$

Show that $\omega = \pm 4dV$ where dV is the volume form on S^3 induced by the standard metric in $\mathbb{R}^4$ (hint: let x_1, x_2, x_3 be some orthonormal basis in $\mathfrak{su}(2)$ with respect to $\mathrm{tr}(a\bar{b}^t)$).

(c) Show that $\frac{1}{8\pi^2}\omega$ is a bi-invariant form on G such that $\left|\frac{1}{8\pi^2}\int_G \omega\right| = 1$ (it only makes sense to talk about absolute value of the integral as there is no preferred orientation on G).