MAT 552: PROBLEM SET 2
DUE TUESDAY 9/28

INSTRUCTOR: ALEXANDER KIRILLOV

---

1. (a) Prove that $\mathbb{R}^3$, considered as Lie algebra with the commutator given by the cross-product, is isomorphic (as a Lie algebra) to $\mathfrak{so}(3,\mathbb{R})$.

(b) Let $\varphi\colon \mathfrak{so}(3,\mathbb{R}) \to \mathbb{R}^3$ be the isomorphism of part (a). Prove that under this isomorphism, the standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ on itself given by the cross-product:

$$a \cdot \vec{v} = \varphi(a) \times \vec{v}, \qquad a \in \mathfrak{so}(3), \vec{v} \in \mathbb{R}^3$$

where $a \cdot \vec{v}$ is the usual multiplication of a matrix by a vector.

2. Write the commutation relations for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in the basis

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3. Write explicitly Lie algebra isomorphisms $\mathfrak{su}(2) \simeq \mathfrak{so}(3,\mathbb{R})$, $(\mathfrak{so}(3,\mathbb{R}))_{\mathbb{C}} \simeq \mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$.

4. Let $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ be the cover map constructed in Problem Set 1.

(a) Show that $\ker\varphi = \{1,-1\} = \{1, e^{\pi i h}\}$, where h is defined in Problem 2.

(b) Using this, show that representations of $\mathrm{SO}(3,\mathbb{R})$ are the same as representations of $\mathfrak{sl}(2,\mathbb{C})$ satisfying $e^{\pi i \rho(h)} = \mathrm{id}$

5. Let P_n be the space of polynomials with real coefficients of degree $\le n$ in variable x. The Lie group $G = \mathbb{R}$ acts on P_n by translations of the argument: $\rho(t)(x) = x + t, t \in G$. Show that the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is given by $\rho(a) = a\partial_x, a \in \mathfrak{g}$ and deduce from this the Taylor formula for polynomials:

$$f(x+t) = \sum_{n\ge 0} \frac{(t\partial_x)^n}{n!} f$$

6. Let $\mathrm{SL}(2,\mathbb{C})$ act on $\mathbf{P}^1$ in the usual way:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x:y) = (ax+by : cx+dy)$$

This defines an action of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ by vector fields on $\mathbf{P}^1$. Write explicitly vector fields corresponding to h, e, f in terms of coordinate $t = x/y$ on the open cell $\mathbb{C} \subset \mathbf{P}^1$.

7. Let J_x, J_y, J_z be the basis in $\mathfrak{so}(3,\mathbb{R})$ described in class. The standard action of $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathfrak{so}(3,\mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation J_x, J_y, J_z for the corresponding vector fields on $\mathbb{R}^3$. Let $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$; this is a second order differential operator on $\mathbb{R}^3$, which is usually called the *spherical Laplace operator*, or the *Laplace operator on the sphere*.

(a) Write Δ_{sph} in terms of $x, y, z, \partial_x, \partial_y, \partial_z$.

(b) Show that Δ_{sph} is well defined as a differential operator on a sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$, i.e., if f is a function on $\mathbb{R}^3$ then $(\Delta_{sph} f)|_{S^2}$ only depends on $f|_{S^2}$.

(c) Show that Δ_{sph} is rotation invariant: for any function f and $g \in \mathrm{SO}(3,\mathbb{R})$, $\Delta_{sph}(gf) = g(\Delta_{sph} f)$.

*(d) Show that the usual Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ can be written in the form $\Delta = \frac{1}{r^2}\Delta_{sph} + \Delta_{radial}$, where Δ_{radial} is a differential operator written in terms of $r = \sqrt{x^2+y^2+z^2}$ and ∂_r.