MAT 534: HOMEWORK 9

DUE TH, NOV 30

Throughout this homework, all vector spaces are considered over the field $\mathbb{F}$.

1. Dummit and Foote, pp. 375-377, exercise 2

2. Dummit and Foote, pp. 375-377, exercise 3

3. Dummit and Foote, pp. 375-377, exercise 9

4. Dummit and Foote, pp. 375-377, exercise 16

5. Dummit and Foote, pp. 375-377, exercise 24

6. Let V_1 and V_2 be subspaces of the same vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1,\ v_2 \in V_2\}$ are also subspaces.

(a) Prove that if V is finite-dimensional, then

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

(b) Show that if V is finite-dimensional, then it is possible to choose a basis $\{v_i\}_{i\in I}$ in V and two subsets $I_1, I_2 \subset I$ such that

- $\{v_i\}_{i\in I_1}$ is a basis of V_1
- $\{v_i\}_{i\in I_2}$ is a basis of V_2
- $\{v_i\}_{i\in I_1\cap I_2}$ is a basis of $V_1 \cap V_2$
- $\{v_i\}_{i\in I_1\cup I_2}$ is a basis of $V_1 + V_2$

7. Let $A\colon V \to V$ be a linear operator on a finite-dimensional space such that $A^2 = A$ (such operators are called idempotent). Prove that then one can write $V = V_1 \oplus V_2$ so that $A|_{V_1} = \mathrm{id}$, $A|_{V_2} = 0$, so A is the projection operator. (Hint: take $V_1 = \operatorname{Im} A$, $V_2 = \operatorname{Ker} A$.)

8. Prove the formula for Vandermonde determinant: if $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, then

$$\det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ & \dots & & & \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} = \prod_{i<j} (\lambda_j - \lambda_i)$$

(Hint: use induction and elementary row and column transformations.)

9. Let A be an operator on a finite-dimensional vector space V. Define the exponent e^A by the following power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

(you can use without proof that this sum is absolutely convergent in the natural topology on the space of linear operators $V \to V$.)

(a) Let P be an invertible operator on V. Prove that $Pe^AP^{-1} = e^{PAP^{-1}}$

(b) Prove that if A and B commute, then $e^{A+B} = e^Ae^B$

(c) Compute the exponent of the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

(d) Prove that if A is antisymmetric (i.e. $A + A^t = 0$), then e^A is orthogonal (i.e. $AA^t = 1$).