MAT 534: HOMEWORK 2

DUE TH, SEPT. 14

Problems marked by asterisk (*) are optional.

1. Let $H \subset G$ be a normal subgroup, and let $f\colon G \to G'$ be a group homomorphism. Show that the map $\bar{f}\colon G/H \to G'$, defined by $[x] \mapsto f(x)$, is well-defined if and only if $H \subset \operatorname{Ker}(f)$; in this case, $\bar{f}$ is a group homomorphism.

(In this situation, we say that f descends to G/H.)

2. Let $A_4 \subset S_4$ be the group of even permutations, and let $H \subset A_4$ be the subgroup generated by elements $x = (12)(34)$, $y = (13)(24)$. Describe the structure of H (i.e., is it isomorphic to a cyclic group? a product of cyclic groups? how large is it). Prove that H is normal in A_4.

3. Prove that any subgroup of index 2 is normal. (Recall that the index of $H \subset G$ is defined as $|G/H|$).

4. Describe all subgroups of symmetric group S_3. For each of them, say whether it is normal; if it is, describe the quotient.

5. Let $\operatorname{Aut}(G)$ be the group of all automorphisms of G, i.e., all isomorphisms $f\colon G \to G$. Prove that $\operatorname{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^\times$, where $\mathbb{Z}_n^\times$ is the group of invertible remainders mod n (with respect to multiplication).

6. Let A, B be groups and let π be an action of B on A by automorphisms: for every $b \in B$, $\pi_b\colon A \to A$ is a group automorphism. Let $G = A \times B$ (as a set) and define on it a binary operation by

$$(a, b)(a', b') = (a\pi_b(a'), bb').$$

Prove that this turns G into a group which is generated by two subgroups $\tilde{A} = \{(a, e_B)\} \simeq A$, $\tilde{B} = \{e_A, b)\} \simeq B$. Moreover, $\tilde{A}$ is normal in G and the composition morphism

$$\tilde{B} \hookrightarrow G \to G/\tilde{A}$$

is an isomorphism.

(So constructed group is called a *semidirect* product: $G = A \rtimes B$)

7. Prove that $\operatorname{Aut}(\mathbb{Z}_8) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and use it to describe all semidirect products $\mathbb{Z}_2 \ltimes \mathbb{Z}_8$ (recall that this means that $\mathbb{Z}_8$ is the normal subgroup). One of these semidirect products is the dihedral group — which one?

8. Let G be a group. For any $g \in G$, let $\varphi_g\colon G \to G$ be the conjugation by g: $\varphi_g(x) = gxg^{-1}$.

(a) Prove that each φ_g is an automorphism of G. (Automorphisms of this form are called *inner automorphisms*).

(b) Prove that $\varphi_g \varphi_h = \varphi_{gh}$. Deduce from it that inner automorphisms form a group, isomorphic to $G/Z(G)$:

$$\operatorname{Inn}(G) \simeq G/Z(G)$$

(Here $Z(G)$ is the center of G.)

(c) Prove that for any (not necessarily inner) automorphism σ, we have $\sigma \circ \varphi_g \circ \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce from this that the group $\mathrm{Inn}(G)$ of inner automorphisms is a normal subgroup in $\mathrm{Aut}(G)$.

***9.** Show that if $G/Z(G)$ is cyclic, then G is Abelian.