MAT 534: HOMEWORK 9

DUE TH, NOV 6

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. (a) Show that if M is an R-module and N — a submodule such that $N, M/N$ are finitely generated, then M is also finitely generated.

(b) Let M be a finitely generated module over a Noetherian ring. Prove that any submodule of M is also finitely generated. [Hint: prove this first for free modules using induction in rank.]

2. A module M over a (not necessarily commutative) unital ring R is called *simple* if it has no nonzero proper submodules.

(a) Prove that every simple module is generated by a single element.

(b) Prove that every simple module is isomorphic to a module of the form R/I, where $I \subset R$ is a maximal left ideal.

(c) Describe all simple modules over $\mathbb{C}[x]$; over $\mathbb{R}[x]$.

3. Dummit and Foote, p. 344, exercise 8.

4. Dummit and Foote, p. 344, exercise 9.

5. Let M be a module over a PID R and $a \in R$ annihilates M: $am = 0$ for any $m \in M$. Assume that $a = a_1 \dots a_n$, where a_i are pairwise relatively prime. Prove that then

$$M = M_1 \oplus \dots \oplus M_n, \qquad M_i = \{m \in M \mid a_i M = 0\}$$

[Hint: first prove it for $n = 2$ and then use induction.]

6. Dummit and Foote, p. 356, exercise 2.

7. Dummit and Foote, p. 469, exercise 11.

8. Dummit and Foote, p. 469, exercise 12.