MAT 534: HOMEWORK 8

DUE TH, OCT 30

1. Complete the argument started in class: for prime p, the polynomial $\dfrac{x^p - 1}{x - 1}$ is irreducible over $\mathbb{Q}$.

2. Dummit and Foote, p. 311, exercise 1

3. Dummit and Foote, p. 311, exercise 2

4. Let $p(x) = x^3 - 2x^2 + 3x - 6 \in \mathbb{Q}[x]$.
  (a) Prove that $\mathbb{Q}[x]/(p)$ is isomorphic to direct product of two fields. Describe these fields.
  (b) Find the inverse of $x + 1$ in $\mathbb{Q}[x]/(p)$, i.e. a polynomial $f \in \mathbb{Q}[x]$ such that $(x + 1)f(x) \equiv 1 \mod p(x)$.

5. Let $\mathbb{F}$ be an algebraically closed field and let $D\colon \mathbb{F}[x] \to \mathbb{F}[x]$ be defined by $D(\sum a_i x^i) = \sum i a_i x^{i-1}$. (This, of course, is the usual definition of derivative :).
  (a) Prove that $D(fg) = D(f) \cdot g + f \cdot D(g)$.
  (b) Prove that if $a \in \mathbb{F}$ is a root of multiplicity k of a polynomial f, then a is also a root of multiplicity $k - 1$ of Df.
  (c) Prove that $f \in \mathbb{F}[x]$ has no multiple roots if and only if $\gcd(f, D(f)) = 1$.

6. Let $\mathbb{F}$ be a field and let

$$f = x^m + a_{m-1}x^{m-1} + \cdots + a_0 \in \mathbb{F}[x]$$
$$g = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in \mathbb{F}[x]$$

be polynomials of degree m, n respectively.
  (a) Prove that the following conditions are equivalent:
    (i) f, g are relatively prime
    (ii) $deg(lcm(f, g)) = m + n$.
    (iii) Collection of $n + m$ polynomials

$$x^i f(x), \qquad i = 0 \dots n - 1$$
$$x^j g(x), \qquad j = 0 \dots m - 1$$

    are linearly independent (over $\mathbb{F}$).
  (b) Prove that there exists a polynomial $R(a_i, b_j)$ in variables $a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1}$ such that f, g are relatively prime iff $R(a_i, b_j) \neq 0$. [This polynomial is called the *resultant* of f, g]. Hint: collection of vectors $v_1, \dots, v_k$ in k-dimensional vector space are linearly independent iff the determinant of the corresponding $k \times k$ matrix is non-zero.

7. Combine two previous problems to prove that if $\mathbb{F}$ is algebraically closed, then $f(x) \in \mathbb{F}[x]$ has multiple roots iff $D = R(f, Df) = 0$. Compute D for $f(x) = x^2 + px + q$.

8. (a) Let $I \subset \mathbb{C}[x, y]$ be the ideal generated by 3 monomials: $x^3; y^3; xy$. Prove that I can not be generated by two elements (not necessarily monomials).
  *(b) (Optional)Prove that for any n, there exists an ideal in $\mathbb{C}[x, y]$ which can not be generated by fewer than n elements.