MAT 534: HOMEWORK 6

DUE TH, OCT 9

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all invertible elements (also called *units*)

(a) $R = \mathbb{F}[x]$

(b) $R = \mathbb{Z}[\omega] \subset \mathbb{C}$, where $\omega \in \mathbb{C}$ is a primitive cubic root of unity.

(c) $R = \mathbb{R}[A] \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$, where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(d) $R = \mathbb{R}[A] \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(e) $R = \mathbb{Z}/n\mathbb{Z}$

2. Let $d \in \mathbb{Z}$, $d > 1$ be squarefree (i.e., d is not divisible by a square of any prime number).

(a) Show that $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\}$ is a field.

(b) Show that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\}$ is a an integral domain.

(c) Define "conjugation" $\overline{} : \mathbb{Q}[\sqrt{d}] \to \mathbb{Q}[\sqrt{d}]$ by $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$. Prove that then $\overline{x+y} = \overline{x} + \overline{y}$, $\overline{xy} = \overline{x} \cdot \overline{y}$.

(d) Show that $u \in \mathbb{Z}[\sqrt{d}]$ is a unit (i.e., has a multiplicative inverse in $\mathbb{Z}[\sqrt{d}]$) iff $u\overline{u} = \pm 1$.

3. Using the previous problem, show that the set of all solutions of the *Pell equation* $a^2 - db^2 = 1$, $a, b \in \mathbb{Z}$, has a structure of an abelian group. Prove that equation $a^2 - 5b^2 = 1$ has infinitely many integer solutions. (Hint: one solution is $(9, 4)$.)

4. Let $\mathbb{F}[[x]]$ be the set of all formal power series in variable x with coefficients in a field $\mathbb{F}$. Prove that $\mathbb{F}[[x]]$ is a ring, and that $a_0 + a_1x + a_2x^2 + \dots$ is a unit in this ring iff $a_0 \neq 0$.

5. Let $p \in \mathbb{R}[x]$ be a quadratic polynomial which has no real roots. Define $R = \mathbb{R}[x]/(p)$.

(a) Show that $R \simeq \mathbb{C}$. [Hint: complete the square]

(b) Show that $R \simeq \mathbb{R}[x, x^{-1}]/(p)$

6. Let $I = (x - y)$, $J = (x + y)$ be ideals in $\mathbb{C}[x, y]$.

(a) Describe explicitly the rings $\mathbb{C}[x,y]/I$, $\mathbb{C}[x,y]/J$, $\mathbb{C}[x,y]/I+J$, $\mathbb{C}[x,y]/IJ$. (Hint: you may make change of variables $x' = x + y, y' = x - y$). Describe each of these rings as polynomial functions on a certain subset in $\mathbb{C}^2$.

(b) Which of the ideals $I, J, I + J, IJ$ is maximal? prime?

7. Let $\mathbb{F}_p$ be the finite field with p elements (p is prime). Compute

(a) the number of one-dimensional subspaces in $\mathbb{F}_p^n$

(b) $|GL_2(\mathbb{F}_p)|$