MAT 534: HOMEWORK 2

DUE TH, SEPT. 11

Problems marked by asterisk (*) are optional.

1. Prove the Third Isomorphism Theorem: if $H, K \subset G$ are normal subgroups, with $K \subset H$, then one has a group isomorphism
$$G/H \simeq \overline{G}/\overline{H}$$
where $\overline{G} = G/K$, $\overline{H} = H/K$.

2. Let $H \subset A_4$ be the subgroup generated by elements $x = (12)(34)$, $y = (13)(24)$. Describe the structure of H (i.e., is it isomorphic to a cyclic group? a product of cyclic groups? how large is it). Prove that H is normal in A_4.

3. Show that the groups S_3, S_4 are solvable.

4. Let $\mathrm{Aut}(G)$ be the group of all automorphisms of G, i.e., all isomorphisms $f\colon G \to G$. Prove that $\mathrm{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^\times$, where $\mathbb{Z}_n^\times$ is the group of invertible remainders mod n (with respect to multiplication).

5. Let A, B be groups and let π be an action of B on A by automorphisms: for every $b \in B$, $\pi_b\colon A \to A$ is a group automorphism. Let $G = A \times B$ (as a set) and define on it a binary operation by
$$(a, b)(a', b') = (a\pi_b(a'), bb').$$
Prove that this turns G into a group which is generated by two subgroups $\tilde{A} = \{(a, e_B)\} \simeq A$, $\tilde{B} = \{e_A, b)\} \simeq B$. Moreover, $\tilde{A}$ is normal in G and the composition morphism
$$\tilde{B} \hookrightarrow G \to G/\tilde{A}$$
is an isomorphism.

(So constructed group is called a *semidirect* product: $G = A \rtimes B$)

6. Prove that $\mathrm{Aut}(\mathbb{Z}_8) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and use it to describe all semidirect products $\mathbb{Z}_2 \ltimes \mathbb{Z}_8$ (recall that this means that $\mathbb{Z}_8$ is the normal subgroup). One of these semidirect products is the dihedral group — which one?

7. Let G be a group. For any $g \in G$, let $\varphi_g\colon G \to G$ be the conjugation by g: $\varphi_g(x) = gxg^{-1}$.

(a) Prove that each φ_g is an automorphism of G. (Automorphisms of this form are called *inner automorphisms*).

(b) Prove that $\varphi_g\varphi_h = \varphi_{gh}$. Deduce from it that inner automorphisms form a group, isomorphic to $G/Z(G)$:
$$\mathrm{Inn}(G) \simeq G/Z(G)$$
(Here $Z(G)$ is the center of G.)

(c) Prove that for any (not necessarily inner) automorphism σ, we have $\sigma \circ \varphi_g \circ \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce from this that the group $\mathrm{Inn}(G)$ of inner automorphisms is a normal subgroup in $\mathrm{Aut}(G)$.

***8.** Show that if $G/Z(G)$ is cyclic, then G is Abelian.