MAT 534: HOMEWORK 12

DUE TH, DEC 4

Unless stated otherwise, all vector spaces are considered over a field $\mathbb{F}$.

1. Find all eigenvectors of the operator A on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}$$

Is it diagonalizable?

2. Let a sequence a_n be defined by

$$a_1 = a_2 = 1, \qquad a_{n+1} = 2a_n + 3a_{n-1}$$

(a) Let

$$X_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} \in \mathbb{R}^2$$

Show that then $X_{n+1} = AX_n$, for some matrix A (not depending on n).

(b) Diagonalize A.

(c) Find a formula for a_n.

3. Let $L\colon V \to V$ be a linear operator. Assume that the ground field $\mathbb{F}$ is algebraically closed.

(a) Prove that L is diagonalizable iff there exists a polynomial p without multiple roots such that $p(L) = 0$.

(b) Let L be diagonalizable and let $W \subset V$ be a subspace which is stable under L: $L(W) \subset W$. Prove that then the restriction of L to W is also diagonalizable.

(c) Prove that if a square matrix A satisfies $A^k = 1$, then it is diagonalizable.

(d) Prove that if a square matrix A satisfies $A^3 = A$, then it is diagonalizable.

4. Let $A, B\colon V \to V$ be commuting operators: $AB = BA$.

(a) Show that if $V_{(\lambda)}$ is the generalized eigenspace for A (that is, $V_{(\lambda)} = \mathrm{Ker}(A-\lambda)^N$ for $N \gg 0$), then $B(V_{(\lambda)}) \subset V_{(\lambda)}$.

(b) Show that if A, B are both diagonalizable, then they can be diagonalized simultaneously: there is a basis in which both A, B are diagonal.

*(c) Diagonalize the operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{x_n + x_2}{2} \\ \frac{x_1 + x_3}{2} \\ \vdots \\ \frac{x_{n-1} + x_1}{2} \end{bmatrix}$$

(hint: this operator commutes with the cyclic permutation of x_i).

5. Use Jordan canonical form to prove that if A is a real $n \times n$ matrix such that all its eigenvalues are real and positive, then there exists a real matrix B such that $B^2 = A$.

6. Dummit and Foote, pg. 499, exercise 7

7. Dummit and Foote, pg. 500, exercise 9

8. Dummit and Foote, pg. 501, exercise 18

9. Construct isomorphisms

(a) $(V \oplus W)^* \simeq V^* \oplus W^*$.

(b) $(V/W)^* \simeq \{f \in V* \mid \langle f, w \rangle = 0 \qquad \forall w \in W\}$.