MAT 534: HOMEWORK 11

DUE TH, NOV 20

Throughout this homework, all vector spaces are considered over the field $\mathbb{F}$.

1. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

2. Let V_1 and V_2 be subspaces of the same vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1; v_2 \in V_2\}$ are also subspaces.

(a) Prove that

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

(b) Show that if V is finite-dimensional, then it is possible to choose a basis $\{v_i\}_{i\in I}$ in V and two subsets $I_1, I_2 \subset I$ such that

- $\{v_i\}_{i\in I_1}$ is a basis of V_1
- $\{v_i\}_{i\in I_2}$ is a basis of V_2
- $\{v_i\}_{i\in I_1\cup I_2}$ is a basis of $V_1 + V_2$

*(c) (optional) Formulate and prove an analog of this for infinite-dimensional case.

3. Let $A\colon V \to V$ be a linear operator on a finite-dimensional space such that $A^2 = A$. Prove that then one can write $V = V_1 \oplus V_2$ so that $A|_{V_1} = \text{id}$, $A_{V_2} = 0$, so A is the projection operator. (Hint: take $V_1 = \operatorname{Im} A$, $V_2 = \operatorname{Ker} A$.)

4. Let A, B be commuting linear operators $V \to V$ such that $A^2 = A$, $B^2 = B$. Prove that then $\operatorname{Ker}(AB) = \operatorname{Ker}(A) + \operatorname{Ker}(B)$

5. Prove the formula for Vandermonde determinant (discussed in class). (Hint: use induction and elementary row and column transformations.)

6. Let A_n be the following $n \times n$ matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

(2 on the diagonal, -1 immediately below and above the diagonal, zeros elsewhere). Use induction to compute the determinant of A_n.

7. Let A be an operator on a finite-dimensional vector space V . Define the exponent e^A by the following power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

(you can use without proof that this sum is absolutely convergent in the natural topology on the space $\operatorname{End}(V)$.)

(a) Let P be an invertible operator on V. Prove that $Pe^AP^{-1} = e^{PAP^{-1}}$

(b) Prove that if A and B commute, then $e^{A+B} = e^Ae^B$

(c) Compute the exponent of the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

(d) Prove that if A is antisymmetric (i.e. $A + A^t = 0$), then e^A is orthogonal (i.e. $AA^t = 1$).