MAT 534: HOMEWORK 9

DUE MON, NOV 19

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Let $p \in \mathbb{R}[x]$ be a quadratic polynomial which has no real roots. Define $R = \mathbb{R}[x]/(p)$.
   (a) Show that $R \simeq \mathbb{C}$. [Hint: complete the square]
   (b) Show that $R \simeq \mathbb{R}[x, x^{-1}]/(p)$

2. Let $I = (x - y)$, $J = (x + y)$ be ideals in $\mathbb{C}[x, y]$.
   (a) Describe explicitly the rings $\mathbb{C}[x, y]/I$, $\mathbb{C}[x, y]/J$, $\mathbb{C}[x, y]/I+J$, $\mathbb{C}[x, y]/IJ$. (Hint: you may make change of variables $x' = x + y, y' = x - y$). Describe each of these rings as polynomial functions on a certain subset in $\mathbb{C}^2$.
   (b) Which of the ideals $I, J, I + J, IJ$ is maximal? prime?

3. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

4. Determine the greatest common divisor in $\mathbb{Q}[x]$ of $a(x) = x^3 + 4x^2 + x - 6$ and $b(x) = x^5 - 6x + 5$ and write it as a linear combination of $a(x)$ and $b(x)$.

5. (a) Prove that every $a \in \mathbb{Z}$ can be uniquely written in the form

$$a = \pm p_1^{n_1} \dots p_k^{n_k} (q_1 \overline{q_1})^{m_1} \dots (q_l \overline{q_l})^{m_l}$$

where $p_i \in \mathbb{Z}$ are integers which are prime(=irreducible) as elements of $\mathbb{Z}[i]$, and $q_i \in \mathbb{Z}[i]$ are irreducible elements of $\mathbb{Z}[i]$ which are not in $\mathbb{Z}$.
   (b) Prove that a prime number $p \in \mathbb{Z}_+$ remains irreducible in $\mathbb{Z}[i]$ iff equation $a^2+b^2 = p$ has no integer solutions. (Hint: $a^2 + b^2 = (a + bi)(a - bi)$.) Deduce from this that prime numbers of the form $4k + 3$ remain irreducible in $\mathbb{Z}[i]$. (In fact, it is known that a prime integer number is irreducible in $\mathbb{Z}[i]$ iff it has the form $4k + 3$.)
   (c) Assuming the statement given in the previous part, prove that for a positive integer n the following statements are equivalent:
   - n can be written as sum of two squares of integer numbers
   - n can be written in the form $n = z\overline{z}$, $z \in \mathbb{Z}[i]$.
   - In the prime factorization for n (in $\mathbb{Z}$), each prime factor of the form $4k+3$ has even exponent.

6. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.
   (a) Prove that elements $2, 3, 1 \pm \sqrt{-5}$ are irreducible in R. [Hint: if $2 = zw$, then $N(z)N(w) = N(2) = 4$, where $N(z) = z\overline{z} \in \mathbb{Z}_+$.]
   (b) Show that R is not unique factorization domain by producing two different factorizations of number 6 into irreducibles in R.