MAT 534: HOMEWORK 3

DUE FRI, SEPT. 21

1. Let $\mathrm{Aut}(G)$ be the group of all automorphisms of G. Prove the following.
(a) $\mathrm{Aut}(\mathbb{Z}_n) = \mathbb{Z}_n^\times$
(b) $\mathrm{Aut}(D_8) = D_8$

2. Prove that $\mathrm{Aut}(\mathbb{Z}_8) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and use it to describe all semidirect products $\mathbb{Z}_2 \ltimes \mathbb{Z}_8$ (recall that this means that $\mathbb{Z}_8$ is the normal subgroup). One of these semidirect products is the dihedral group — which one?

3. Let $H \subset A_4$ be the subgroup generated by elements $x = (12)(34)$, $y = (13)(24)$. Describe the structure of H (i.e., is it isomorphic to a cyclic group? a product of cyclic groups? how large is it). Prove that H is normal in A_4.

4. Show that the groups S_3, S_4 are solvable.

5. (a) Let p be a prime number. Classify all groups of order p.
(b) Classify all groups of order 6.
(c) Let p and q be different prime numbers. Classify all Abelian groups of order pq.

6. Let p and q be primes (not necessarily distinct) with $p \le q$. Prove that if p does not divide $q - 1$, then any group of order pq is Abelian.

Hint: Using the class equation, prove that any noncommutative group G of order pq has an element of order q. This element generate the normal cyclic subgroup H of order q. Study the action of G on H by conjugations and compare the resulting automorhisms of H with the possible automorphisms of a cyclic group of order q.

7. Describe all Sylow 2-subgroups and 3-subgroups of D_6 (symmetries of a regular hexagon).

8. Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.