MAT 511: PRACTICE MIDTERM SOLUTIONS

TU, OCT 11, 2016

Your name:______________________________________
(please print)

This is a practice midterm. It is longer and the actual one — to give you more practice.
No books or notes other than the handout posted on course web page.
Notation:
$\mathbb{Z}$ — integer numbers
$\mathbb{Z}_+$ — positive integers
$\mathbb{R}$ — real numbers

1. A (fictional) tax form contains the following statement:

You can use the tax deduction if you do not itemize deductions and your adjusted gross income is less than \$54, 000 (\$61, 000 for married filing jointly).

Write this statement in the language of propositional logic using notation

D: you can use the tax deduction

I: you itemize deductions

M: your filing status is "married filing jointly"

P: your adjusted gross income is less than \$54, 000

Q: your adjusted gross income is less than \$61, 000

Solution: This can be written as

$$\Big(\sim I \wedge (\text{your income is lower than the cutoff})\Big) \implies D$$

The income condition can be written as $P \vee (M \wedge Q)$. Thus, the answer is

$$\Big(\sim I \wedge (P \vee (M \wedge Q))\Big) \implies D$$

Note: the income condition can be also written as $(\sim M \wedge P) \vee (M \wedge Q)$. Both forms are correct (in fact, they are equivalent.)

2. Consider the following statement:

"Any student in this class who gets an A for the midterm is either a genius or a hard worker".

(a) Rewrite this statement using quantifiers, logical connectives, and the following notation:

S — set of all students in the class
$A(x)$: student x gets an A for the midterm.
$G(x)$: student x is a genius
$H(x)$: student x is a hard worker

Solution:

$$\forall x \in S : A(x) \implies (G(x) \vee H(x))$$

(b) Write the negation of this statement, both in formal language using quantifiers and in plain English.

Solution:

$$\sim \Big(\forall x \in S\colon A(x) \implies (G(x) \vee H(x))\Big)$$

Using the laws of logic, we can rewrite this as

$$\exists x \in S\colon \quad \sim (A(x) \implies (G(x) \vee H(x)))$$

or, equivalently

$$\exists x \in S\colon (A(x) \wedge \sim (G(x) \vee H(x))$$

In plain English: some students in the class get A for the midterm without being geniuses or working hard.

3. Prove that for any $n \geq 3$, we have

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\geq\frac{3}{5}$$

Solution: proof by induction.

Base case: $n=3$. In this case, the inequality becomes

$$\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\geq\frac{3}{5}$$

Since the left-hand side is equal to

$$\frac{15}{60}+\frac{12}{60}+\frac{10}{60}=\frac{37}{60}$$

and $\frac{3}{5}=\frac{36}{60}$, the inequality is true.

Induction step: denote for simplicity

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=a_n.$$

Assume that $a_n \geq 3/5$. We need to prove that then, $a_{n+1}\geq 3/5$. But

$$\begin{aligned}a_{n+1}&=\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n+2}\\&=a_n-\frac{1}{n+1}+\frac{1}{2n+1}+\frac{1}{2n+2}\end{aligned}$$

Since

$$-\frac{1}{n+1}+\frac{1}{2n+1}+\frac{1}{2n+2}\geq-\frac{1}{n+1}+\frac{1}{2n+2}+\frac{1}{2n+2}=-\frac{1}{n+1}+\frac{2}{2n+2}=0$$

we see that $a_{n+1}\geq a_n$. Thus, if $a_n\geq 3/5$, then $a_{n+1}\geq 3/5$. This completes the proof.

4. Prove the following statement

$$\forall x \in \mathbb{R} - \{0\} : \ x + \tfrac{1}{x} > 0 \implies x > 0$$

You can use all the usual properties of real numbers, including properties of inequalities.

Solution: Let $x \in \mathbb{R} - \{0\}$. We need to prove that then

$$x + \tfrac{1}{x} > 0 \implies x > 0 \tag{1}$$

Since $x \neq 0$, either $x > 0$ or $x < 0$. Consider these two cases separately.

Case 1: $x > 0$. Then $x + \frac{1}{x} > 0$ is true, and $x > 0$ is also true, so (1) is true.

Case 2: $x < 0$. Then $\frac{1}{x} < 0$, so $x + \frac{1}{x} < 0$. Thus, in this case (1) becomes $F \implies F$, which is true. So, in this case (1) is also true.

5. For each of the following statements tell whether it is true or false, and justify your answer by giving a proof. You can use all the usual properties of real numbers, including properties of inequalities.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 0 < y < x$ TRUE FALSE

Solution: FALSE. Counterexample: take $x = -1$.

(b) $\forall x \in \mathbb{R}_+, \exists y \in \mathbb{R}_+ : 0 < y < x$ TRUE FALSE

(here $\mathbb{R}_+$ is the set of positive real numbers)

Solution: TRUE. Given $x > 0$, take $y = x/2$. Then $0 < y < x$.

(c) $\exists y \in \mathbb{R}_+, \forall x \in \mathbb{R}_+ : 0 < y < x$ TRUE FALSE

(here $\mathbb{R}_+$ is the set of positive real numbers)

Solution: FALSE. Assume that such a y exists; then for any $x \in \mathbb{R}_+$, we have $0 < y < x$. Taking $x = y$, we get a contradiction.

6. Prove that for any sets A, B we have $A \cap (A \cup B) = A$. Note: you can use Venn diagrams to illustrate your proof; however, Venn diagrams by themselves will not be considered sufficient proof.

Solution: Assume $x \in A \cap (A \cup B)$. Then, by definition of intersection, $x \in A$ and $x \in A \cup B$. This implies $x \in A$.

Conversely, assume $x \in A$. Then, by definition of union, $x \in A \cup B$. Thus, $(x \in A) \wedge (x \in A \cup B)$, so $x \in A \cap (A \cup B)$.

Therefore, we have shown that $(x \in A) \iff \Big(x \in A \cap (A \cup B)\Big)$. By definition, it means that $A \cap (A \cup B) = A$.