MAT 511: HOMEWORK 8

DUE TH, NOV 3

1. Let $A = \mathbb{R}\times\mathbb{R} - \{(0,0)\}$ be the plane with the origin removed. Consider the following relation on A:

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{if } \exists t \in \mathbb{R}_{>0} \colon x_2 = tx_1, y_2 = ty_1$$

(this is the equivalence relation from problem 2 of the previous HW).

Show that the set of equivalence classes $A/\sim$ can be identified with the unit circle S in $\mathbb{R}^2$: it is possible to construct a function $(A/\sim) \to S$, assigning to an equivalence class a point on the circle, and this function will be one-to-one and onto.

2. Consider the equivalence relation on $\mathbb{R}$ given by $a \sim b$ if $a - b$ is an integer multiple of 360. Show that the set of equivalence classes $\mathbb{R}/\sim$ can be identified with the unit circle in $\mathbb{R}^2$.

3. Let $\mathbb{Z}_n = \mathbb{Z}/(\equiv \mod n)$ be the set of congruence classes mod n, i.e. the set of equivalence classes for the relation $a \equiv b \mod n$.

(a) Compute $[2]^{20}$ in $\mathbb{Z}_5$. [Hint: start by computing $[2]^2$, $[2]^3$, $\dots$]

(b) Show that the function $f \colon \mathbb{Z}_5 \to \mathbb{Z}_8$ given by $f([a]) = [2a]$ is not well-defined: for an equivalence class $x \in \mathbb{Z}_5$, $f(x)$ depends on the choice of representative in x.

(c) Show that the function $\mathbb{Z}_{10} \to \mathbb{Z}_5$ given by $g([a]) = [a]$ is well defined. Find its range.

(d) For the function g from the last problem, find the preimage $g^{-1}(\{[1], [2]\})$.

4. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sin x, \qquad g(x) = 2x^2 + 1.$$

Compute $f \circ g$, $g \circ f$. Find the range of each of these compositions. (You can use all the properties of quadratic and trigonometric functions you had learned in algebra or precalculus)

5. Let $f \colon R \to \mathbb{R}$ be given by $f(x) = x^2 + 1$. Find

(a) $f([2,3])$

(b) $f([-2,2])$

(c) $f^{-1}([2,3])$

(d) $f^{-1}([-1,1])$

(e) $f^{-1}([-2,-1])$

(You can use all the properties of quadratic and trigonometric functions you had learned in algebra or precalculus)

6. Let $f \colon A \to B$ be a function and let $X_1, X_2 \subseteq A$.

(a) Prove that $f(X_1) \cup f(X_2) = f(X_1 \cup X_2)$

(b) Give a counterexample showing that $f(X_1) \cap f(X_2)$ is not necessarily equal to $f(X_1 \cap X_2)$