MAT 319/320: HOMEWORK 7

DUE FRIDAY, NOV 3

1. Let the sequence a_n be defined by $a_1 = 2$, $a_{n+1} = 2 + \frac{1}{a_n}$. Show that a_n is converging and compute the limit. [Hint: use Theorem 3.5.8]

2. For each of the following series, find whether it is convergent. If possible, find the sum.

(a) $\sum_{n\geq 1} \dfrac{1}{(n+1/2)(n-1/2}$

(b) $\sum_{n\geq 1} \dfrac{1}{5n+1}$

(c) $\sum_{n\geq 1} \dfrac{n^2-1}{n^2+1}$

(d) $\sum_{n\geq 1} \dfrac{1}{n^3-2}$ [do not try to find the sum]

(e) $\sum_{n\geq 1} \dfrac{\sin(n\pi/3)}{2^n}$ [hint: computing the sum is only possible using complex numbers, even though the answer is real.]

(f) $\sum_{n\geq 1} \dfrac{n^2}{n!}$ [do not try to find the sum]

3. For a series $\sum a_n$, let us consider two "subseries": sum of even terms $\sum a_{2n}$ and sum of odd terms $\sum a_{2n-1}$.

(a) Use Cauchy criterion to show that if both $\sum a_{2n}$, $\sum a_{2n-1}$ converge, then $\sum a_n$ also converges, and $\sum a_n = \sum a_{2n} + \sum a_{2n-1}$.

(b) Give an example when the converse statement fails, i.e. $\sum a_n$ converges but $\sum a_{2n}$, $\sum a_{2n-1}$ diverge. [Hint: take a_n to be an alternating sequence: $a_n > 0$ for n odd and $a_n < 0$ for n even.]

4. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^3 - 3$.

(a) For $\varepsilon = 0.1$, find δ such that $|f(x) - 13| < \varepsilon$ for all $x \in V_\delta(2)$.

(b) Prove from the definition that $\lim_{x\to 2} f(x) = 13$.

5. Let $f\colon (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$$

Do the limits $\lim_{x\to 1} f(x)$, $\lim_{x\to 0} f(x)$ exist? If so compute them.