MAT 319/320: HOMEWORK 6

DUE FRIDAY, OCT 27

Some of the problems below use the notion of cluster point of a sequence. Since it is not covered in the textbook (but was covered in the lecture), we repeat the definition here:

Definition. A point p is called a *cluster point* of the sequence a_n if in every neighborhood of p there are infinitely many terms of the sequence:

$$\forall \varepsilon > 0 \text{ the set } \{n \in \mathbb{N} \mid a_n \in V_\varepsilon(p)\} \text{ is infinite}$$

Similarly, a point p is called a *cluster point* of a set S if in any neighborhood of p there are infinitely many elements of S.

1. (a) Give an example of a sequence that is bounded above but not bounded below and that has a convergent subsequence.
(b) Explain how to construct a monotone increasing sequence of rational numbers that converges to $\sqrt{3}$.

2. Let sequence a_n and point p be such that in any neighborhood of p there is at least one term of the sequence distinct from p itself. Show that then p is a cluster point of a_n.

3. Let a_n be the following sequence:

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$$

(this sequence contains all rational numbers in the interval $(0, 1)$; we have used similar sequences to prove countability of $\mathbb{Q}$).
Find all cluster points of a_n.

4. If $0 < r < 1$ and sequence a_n is such that $|a_{n+1} - a_n| < Cr^n$ (where C is some constant independent of n), then a_n converges.

5. Show that the Cauchy Criterion fails for rational numbers: it is possible to have a Cauchy sequence of rational numbers which does not converge to a rational number.

6. Show that if a_n is an unbounded sequence, then there exists a properly divergent subsequence.

7. For each of the following sequences, determine whether it is convergent, properly divergent, or neither. (You do not have to find the limit).
(a) $\sqrt{n}$
(b) $\sqrt{n^2 + 2}$
(c) $\sin(\sqrt{n}\pi)$
(d) $\dfrac{\sin(\sqrt{n}\pi)}{\sqrt{n}}$
(e) $\dfrac{\sqrt{n^2 + 2}}{n}$