MAT 319/320: HOMEWORK 2

DUE FRIDAY, SEPT. 22

In problems involving real numbers, please make sure that you are only using axioms of real numbers and results proved in the book. Give precise references, e.g. "by Theorem 2.1.7".

1. Recall that an integer $n \geq 2$ is called prime if it has no divisors other than 1 and itself. Prove, using full induction, that any positive integer ≥ 2 can be written as a product of prime numbers.

2. (a) List all the subsets of $S = \{1, 2\}$ (do not forget $\varnothing$ and S itself, both of which are subsets of S).
  (b) List all the subsets of $S = \{1, 2, 3\}$. (Try to see a relation with (a) — this should give you an idea for the inductive step below.)
  (c) Prove that for all $n \geq 1$, if a finite set S has n elements then it has 2^n different subsets. [Hint: Use induction. You should have checked this above for $n = 2$, 3, but you need to do the inductive step.]

3. Show that the set $\mathbb{Z}_{odd}$ of odd (positive and negative) integers is denumerable by
  (a) enumerating them and
  (b) giving an explicit formula for the corresponding bijection $f\colon \mathbb{Z}_{odd} \to \mathbb{N}$.

4. Let $a, b \in \mathbb{R}$ be such that $0 < a < b$. Prove by induction that for any positive integer n, $a^n < b^n$.

5. Let $a, b \in \mathbb{R}$. Show that $a^2 + b^2 = 0$ if and only if $a = 0$ and $b = 0$. [Hint: use order relation.]

6. Let $a, b, x, y \in \mathbb{R}$ be such that $a < x < b$, $a < y < b$. Show that then $|x - y| < b - a$.

7. Let $a, b \in \mathbb{R}$, $a \neq b$. Show that there exist ε-neighborhoods of a, b which do not intersect: $V_\varepsilon(a) \cap V_\varepsilon(b) = \varnothing$.

Bonus problem. Show that the set of all rational numbers is enumerable.