MAT 319/320: HOMEWORK 1

DUE FRIDAY, SEPT. 15

1. Prove the identity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

2. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x$
(a) Graph it.
(b) Find $f(A)$ where A is the open interval $(0, 4)$.
(c) Find $f^{-1}(B)$ where $B = [1, 4]$.
(d) Find two subsets C, D of $\mathbb{R}$ such that $f(C) \cap f(D) \neq f(C \cap D)$.

3. Let $f\colon A \to B$ be a function and suppose that $C \subseteq A$ and $D \subseteq B$. Are the following statements true or false (for every choice of f, C, D)? Justify your answers by a brief proof or a counterexample.
(a) $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
(b) $f^{-1}(B \setminus D) = f^{-1}(B) \setminus f^{-1}(D)$.
Hint: as in question 2, try some examples. You can try functions $f\colon \mathbb{R} \to \mathbb{R}$ or you can try functions $f\colon A \to B$ where A and B are finite sets.

4. Suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions such that the composition $g \circ f$ is surjective. Is g necessarily surjective? What about f? Give brief proofs or counterexamples.

5. Prove that for any positive integer n,

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

6. Prove by induction that for any $n \geq 5$, $n^2 < 2^n$. [Hint: prove first that $(n+1)^2 < 2n^2$.]

7. Guess a general formula for the product

$$\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{9}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

and prove it by induction.

8. Let the Fibonacci sequence be defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. Prove that then

$$F_n F_{n+1} = F_1^2 + F_2^2 + \cdots + F_n^2$$