# MAT 319 HANDOUT 1: LIMITS OF SEQUENCES

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## 1. Basic theorems about limits

We say that the sequence $a_n$ converges (or is convergent) if the limit $\lim a_n$ exists and is finite.

**Theorem 1.** *If a sequence converges, then it is bounded.*

**Theorem 2.** *If the sequence $a_n$ converges and $a_n \geq 0$ for all $n$, then $\lim a_n \geq 0$.*

Note: if we replace $\geq$ by $>$ in both places, the statement could fail: there are positiive sequences whose limit is equal to zero.

**Theorem 3** (Comparison theorem)**.** *If $\lim a_n = 0$ and $|b_n| \leq |a_n|$ for all $n$, then $\lim b_n = 0$.*

**Theorem 4** (Sum, product, and quotient rule for limits)**.** *If sequences $a_n, b_n$ converge, then*

**1.** $\lim(k \cdot a_n) = k \lim a_n$
**2.** $\lim(a_n + b_n) = (\lim a_n) + (\lim b_n)$
**3.** $\lim(a_n b_n) = (\lim a_n) \cdot (\lim b_n)$
**4.** *If, in addition, $\lim b_n \neq 0$, then $\lim(a_n/b_n) = (\lim a_n)/(\lim b_n)$.*

Some (but not all) of these results also apply when one or both limits are infinite. See Section 9 in the book.

## 2. Existence of limits

The results below are based ont he use of Completeness Axiom for real numbers.

**Theorem 5** (Monotone convergence)**.** *Every bounded above increasing sequence has a limit; moreover, for such a sequence $\lim a_n = \sup a_n$.*

A similar result also holds for decreasing sequences.

**Theorem 6** (Nested intervals property)**.** *Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ... be a sequence of nested intervals:*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

*Then there exists a common point: $\exists c \in \mathbb{R} : c \in [a_k, b_k]$ for all $k$.*

Note: this theorem is not in the book!

**Definition.** A sequence is called a *Cauchy sequence* if the following property holds:

$$\forall \varepsilon > 0 \ \exists N \colon k \geq N, l \geq N \implies |a_k - a_l| < \varepsilon$$

**Theorem 7.** *A sequence converges if and only if it is a Cauchy sequence.*

## 3. Subsequences

**Theorem 8.** *If* $\lim a_n = L$ *(finite or infinite), then for any subsequence* $t_n$ *of* $a_n$*, we have* $\lim t_n = L$*.*

**Theorem 9.** *A number* $A$ *is a limit of some subsequence of* $a_n$ *if and only if, for every* $\varepsilon > 0$*, the interval* $(A - \varepsilon, A + \varepsilon)$ *contains infinitely many terms of the sequence.*

**Theorem 10** (Bolzano–Weierstrass)**.** *Any bounded sequence contains a convergent subsequence.*

## 4. Lim sup, lim inf

For a sequence $s_n$, denote by $S$ the set of subsequential limits of $S$, i.e. the set of limits of all possible subsequences of $s_n$ (including infinite limits):

$$S = \{L \mid L = \lim t_n \text{ for some subsequence } t_n \text{ of } s_n\}$$

This set is always non-empty: for a bounded sequence, by Bolzano-Weierstrass theorem; for unbounded sequence, you can find a subsequence with limit $\pm\infty$.

**Definition.** Let $s_n$ be a sequence and let $S$ be the set of subsequential limits of $s_n$ as above. Then we define

$$\limsup(s_n) = \sup(S)$$
$$\liminf(s_n) = \inf(S)$$

These numbers are defined for any sequence $s_n$.

**Theorem 11.** *A sequence* $s_n$ *has a limit (finite or infinite) if and only if* $\limsup s_n = \liminf s_n$*. In this case,* $\lim s_n = \limsup s_n = \liminf s_n$*.*

**Theorem 12.** $\limsup s_n$ *is itself a limit of some subsequence of* $s_n$*. Thus,* $\limsup s_n = \max(S)$*.*

A similar result holds for $\liminf$.

The result below was mentioned in class, but was not fully proved.

**Theorem 13.** *If* $A = \limsup s_n$ *is finite, then for any* $\varepsilon > 0$ *there are only finitely many terms of the sequence satisfying* $s_n > A + \varepsilon$*, and there are infinitely many terms of* $s_n$ *satisfying* $s_n > A - \varepsilon$*.*