Throughout this problem set, \( \mathbb{F} \) is a field.

1. Let \( \alpha = \sqrt{2} + \sqrt{3} \in \mathbb{R} \).
   
   (a) Show that \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \).
   
   (b) Find the minimal polynomial of \( \alpha \) over \( \mathbb{Q} \).
   
   (c) Let \( \alpha' = \sqrt{2} - \sqrt{3} \). Show that there exists a field isomorphism \( \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha') \) which sends \( \alpha \) to \( \alpha' \).

2. For each of the following polynomials, describe its splitting field over \( \mathbb{Q} \).
   
   (a) \( x^4 + 1 \)
   
   (b) \( x^3 - 5 \)
   
   (c) \( x^4 + x^2 + 1 \)

3. Let \( \mathbb{F} \) be a field of characteristic zero.

   For a polynomial \( f = \sum a_k x^k \in \mathbb{F}[x] \), define its derivative by
   
   \[ Df = \sum ka_k x^{k-1} \in \mathbb{F}[x]. \]

   (a) Show that the derivative satisfies familiar rules:
   
   \[ D(f + g) = Df + Dg, \quad D(fg) = (Df)g + f(Dg). \]

   (b) Show that if \( \mathbb{E} \supset \mathbb{F} \) is an extension of \( \mathbb{F} \), and \( a \in \mathbb{E} \) is a root of \( f \) of order \( m \geq 1 \), then \( a \) is a root of \( Df \) of order \( m - 1 \). Is this true if \( \mathbb{F} \) has positive characteristic?

   (c) Show that \( f \) has no multiple roots (in any extensions of \( \mathbb{F} \)) iff \( \gcd(f, Df) = 1 \). In particular, it holds if \( f \) is irreducible.

4. Let \( \mathbb{F} \) be a field of characteristic \( p > 0 \).

   (a) Show that the map \( Fr: \mathbb{F} \to \mathbb{F} \) given by \( Fr(x) = x^p \) is a homomorphism of fields. Deduce from this that if \( \mathbb{F} \) is finite, then \( Fr \) is a bijection. [It is called the Frobenius automorphism].

   (b) Show that the the set \( \{ x \in \mathbb{F} \mid x^p = x \} \) is a subfield in \( \mathbb{F} \), which is isomorphic to \( \mathbb{Z}_p \). [Hint: how many different roots does the polynomial \( x^p - x \) have?]