MAT 314: HOMEWORK 4

DUE TH, MARCH 7, 2019

Throughout the assignment, R is a principal ideal domain. All modules are assumed to be finitley generated and with finite set of relations.

Most questions will be about the structure theorem: every such module is isomorphic to a module of the form

$$M \simeq R^r \oplus \Big(\bigoplus_i R/(p_i^{k_i})\Big) \tag{1}$$

where p_i are irreducible (not necessarily distinct).

1. Let $R = \mathbb{R}[x]$, $M = R/(x^3 + x - 10)$. Write M in the form (1).

2. Let M be a module over R and let $a \in R$, $a \neq 0$ be such that $am = 0$ for any element $m \in M$.

(a) Show that all irreducibles p_i appearing in the canonical form (1) of M must be divisors of a. [Hint: see problem 1 of the previous assignment.]

(b) Show that if $a = q_1 \dots q_m$, where q_i are *distinct* irreducibles, then all powers k_i appearing in the canonical form (1) of M must be 1.

3. Use the previous problem to show that if $A\colon V \to V$ is a linear operator in a finite-dimensional vector space over $\mathbb{C}$, and $A^k = I$ for some k, then A is diagonalizable. Is the same true over $\mathbb{R}$?

4. Let $R = \mathbb{C}[x]$, $M = (V, A)$ (as discussed many times before). Define

$$V^{(\lambda)} = \{v \in V \mid (A - \lambda)^k v = 0 \text{ for large enough } k\}$$

This is commonly called *generalized eigenspace.*

(a) Prove that each $V^{(\lambda)}$ is a subspace which is stable under A: $AV^{(\lambda)} \subset V^{(\lambda)}$

(b) Prove that $V^{(\lambda)}$ is the direct sum of all Jordan blocks with eigenvalue λ on the diagonal.

(c) Prove that $V = \bigoplus_\lambda V^{(\lambda)}$ (direct sum over all eigenvalues).

***5.** Can you formulate and then prove an analog of the previous problem for a module over arbitrary PID R?