Throughout the assignment, words PID mean “principal ideal domain”. You can use all the results about PIDs discussed in class, in particular:

- \( \gcd(a, b) = d \iff (a) + (b) = (d) \)
- \( \lcm(a, b) = m \iff (a) \cap (b) = (m) \)
- \( \gcd(a, b) = 1 \iff a \) is invertible in \( R/(b) \)

1. Let \( R \) be a PID, and \( a, b \in R \) be such that \( \gcd(a, b) = 1 \).
   (a) Prove that \( ax \) is divisible by \( b \) iff \( x \) is divisible by \( b \).
   (b) Prove that then \( \lcm(a, b) = ab \).

2. Determine the greatest common divisor in \( \mathbb{Q}[x] \) of \( a(x) = x^4 - 1 \) and \( b(x) = x^5 - 1 \) and write it as a linear combination of \( a(x) \) and \( b(x) \).

3. Let \( R \) be a PID.
   (a) Let \( I_1 \subset I_2 \subset I_3 \cdots \subset R \) be a sequence of ideals. Prove that it stabilizes: for large enough \( k \), \( I_k = I_{k+1} = I_{k+2} \ldots \). [Hint: consider \( I = \bigcup I_k \); then it is an ideal and thus must be generated by a single element.]
   (b) Let \( a_1, a_2, \cdots \in R \) be a sequence of non-zero elements such that \( a_{k+1} \) is a proper divisor of \( a_k \) (i.e., \( a_k/a_{k+1} \) is not invertible). Prove that this sequence can not be infinite.

4. Consider the ring \( R = \mathbb{Z}[\sqrt{-5}] = \{ a + bi\sqrt{5} \mid a, b \in \mathbb{Z} \} \subset \mathbb{C} \). For an element \( z = a + ib\sqrt{5} \in R \), we denote \( N(z) = zz = a^2 + 5b^2 \in \mathbb{Z} \).
   (a) Show that \( N(zw) = N(z)N(w) \).
   (b) Show that if \( N(z) = 1 \), then \( z = \pm 1 \).
   (c) Prove that there are no elements \( z \in R \) with \( N(z) = 2 \).
   (d) Prove that elements \( 2, 3, 1 \pm \sqrt{-5} \) are irreducible in \( R \). [Hint: if \( 2 = zw \), then \( N(z)N(w) = N(2) = 4 \).]
   (e) Show that \( 6 \in R \) admits two different factorizations into irreducibles in \( R \). [Thus, \( R \) is not a unique factorization domain and thus can not be a PID.]