Throughout this problem set, $F$ is a field of characteristic zero.

1. Let $p(x) \in F[x]$ be a polynomial of degree $n$. Let $L$ be the splitting field of $p$, and let $x_1, \ldots, x_n \in L$ be the roots of $p$. Define
\[
\Delta = \prod_{i<j} (x_i - x_j) \in L
\]
(a) Prove that for every $g \in \text{Gal}(L/F)$, we have $g(\Delta) = \text{sgn}(g)\Delta$, where $\text{sgn}(g)$ is the sign of the corresponding permutation (recall that $\text{Gal}(L/F) \subset S_n$).
(b) Prove that $D = \Delta^2 \in F$ ($D$ is called the discriminant of $p$).
(c) Prove that $\Delta \in F$ iff $\text{Gal}(L/F) \subset A_n$ (where $A_n$ is the subgroup of even permutations).

2. Describe the Galois groups of the following polynomials over $\mathbb{Q}$:
   (a) $x^3 - 3x + 1$
   (b) $x^3 - 3x + 3$
   (You can use without proof the result I quoted in class: for a cubic polynomial $x^3 + px + q$, the discriminant is $D = -4p^3 - 27q^2$.)

3. Let $G$ be a group, with subgroups $G_1, G_2 \subset G$ such that $G_1$ is a normal subgroup of $G_2$. Let $\varphi : G \to H$ be a group homomorphism.
   (a) Prove that $\varphi(G_1)$ is a normal subgroup of $\varphi(G_2)$.
   (b) Prove that if $G_2/G_1$ is commutative, then so is $\varphi(G_2)/\varphi(G_1)$.

4. A group $G$ is called \textit{solvable} if there exists a finite collection of subgroups
\[
\{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G
\]
such that each $G_i$ is normal subgroup of $G_{i+1}$ and the quotient $G_{i+1}/G_i$ is commutative.
   (a) Prove that $S_3, S_4$ are solvable
   (b) Use the previous problem to prove that if $G$ is solvable, then any quotient of $G$ is also solvable.