Throughout this assignment, $R$ is an arbitrary associative ring with unit (not necessarily commutative). All modules are modules over $R$. Letter $F$ always stands for a field.

1. As discussed in class, a module over the ring $\mathbb{F}[x]$ can be described as a pair $(V, A)$, where $V$ is a vector space over $\mathbb{F}$ and $A: V \rightarrow V$ is a linear operator.
   (a) Give such a description for the module $M = \mathbb{F}[x]/(x^2 + 2x + 2)$, by constructing an explicit basis in the corresponding vector space $V$ and writing the operator $A$ as a matrix in this basis.
   (b) Show that for $\mathbb{F} = \mathbb{C}$, the module $M$ constructed in the previous part is isomorphic to a direct sum $M_1 \oplus M_2$, where $M_1, M_2$ are $\mathbb{C}[x]$ modules which are one-dimensional vector spaces over $\mathbb{C}$. [Hint: is $A$ diagonalizable?]. Is the same true when $\mathbb{F} = \mathbb{R}$?

2. Let $M$ be a an $R$-module and $N \subset M$, an $R$-submodule. Let $M' = M/N$ and denote by $f: M \rightarrow M'$ the obvious homomorphism of modules.
   (a) Show that if $K' \subset M'$ is a submodule, then $K = f^{-1}(K') \subset M$ is also a submodule, and $M/K$ is isomorphic to $M'/K'$.
   (b) Show that the construction of the previous part gives a bijection
      $$(\text{Submodules } K' \subset M/N) \leftrightarrow (\text{Submodules } K \subset M \text{ which contain } N)$$

3. Let $M$ be an $R$-module and $M_1, M_2 \subset M$ be submodules such that $M = M_1 + M_2$, i.e. every element in $M$ can be written as a sum $m = m_1 + m_2$, $m_1 \in M_1, m_2 \in M_2$ (possibly not uniquely). Prove that then one has an isomorphism $M \simeq (M_1 \oplus M_2)/(M_1 \cap M_2)$.

4. A module $M$ over $R$ is called simple if it has no nonzero proper submodules.
   (a) Prove that every simple module is generated by a single element.
   (b) Prove that every simple module is isomorphic to a module of the form $R/I$, where $I \subset R$ is a maximal left ideal.
   (c) Describe all simple modules over $\mathbb{Z}$ (i.e., abelian groups).
   *(d) Describe all simple modules over $\mathbb{C}[x]$.

5. Let $R = \text{Mat}_{n \times n}(\mathbb{F})$ be the ring of $n \times n$ matrices with entries in a field $F$. Then $\mathbb{F}^n$ is naturally a module over $R$. Show that it is simple.
   Hint: show that for any nonzero vector $v \in \mathbb{F}^n$, the subspace $Rv$ contains the basis vector
   $$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$