MAT310 Fall 2012
Practice Midterm I

The actual midterm will consist of six problems.

Problem 1 If U and W are subspaces of a linear space:F, show that $U \cup W$ need not be a subspace. However, if $U \cup W$ is a subspace, show that either $U \subset W$ or $W \subset U$.

Problem 2

- Show that the set $\{1, (t-1), (t-1)^2, (t-1)^3\}$ generates $P_3(\mathbb{R})$.
- Can two disjoint subsets of $\mathbb{R}^2$, each containing two vectors, have the same span? Explain.

Problem 3 Let $V \stackrel{\phi}{\to} W \stackrel{\psi}{\to} V$ be linear maps such that $\psi\phi : V \to V$ is an isomorphism. Show that ϕ is one-to-one (injective) and ψ is onto (surjective).

Problem 4 Let $V \overset{\phi}{\to} W$ be a linear map of finite-dimensional linear spaces and let $L \subset V$ be a linear subspace.

- Show that dimension of $\phi(L) = \{w \in W | w = \phi(v), v \in V\}$ is not greater then dimension of L.
- What is the relation between $\dim \phi(L)$ and $\dim L$ when ϕ is one-to-one.

Problem 5 A linear map $\rho : V \to V$ is idempotent if $\rho\rho = \rho$. Show that if ρ is idempotent then ρ acts as the identity on range(V). (Such linear maps are called projections: ρ projects V onto range(V).)

Problem 6 Determine whether or not $\{(1, 1, 0), (2, 0, -1), (-3, 1, 1)\}$ is a basis for $\mathbb{R}^3$.

Problem 7 $\psi : V \to V$ is nilpotent of order 2 if ψ^2 is the zero endomorphism. Now composition of two such endomorphisms need not be nilpotent of order 2. Find $\psi, \phi : \mathbb{R}^2 \to \mathbb{R}^2$, each nilpotent of order 2, whose composition is idempotent.

Problem 8 If x and y are vectors and M is a subspace of V such that $x \notin M$ but $x \in \text{span}\{M, y\}$, does it follow that

$$\text{span}\{M, y\} = \text{span}\{M, x\}$$

Problem 9 Is it true that if L, M, and N are subspaces of a vector space, then

$$L \cap (M + (L \cap N)) = (L \cap M) + (L \cap N)?$$

Problem 10

1. Under what conditions on the scalars $\alpha, \beta \in \mathbb{C}$ are the vectors $(1, \alpha)$ and $(1, \beta)$ in $\mathbb{C}^2$ linearly independent?
2. Is there a set of three linearly independent vectors in $\mathbb{C}^2$ considered as a vector space over
   (a) Real numbers
   (b) Complex numbers.
3. Under what conditions on the scalar $x \in \mathbb{C}$ do the vectors $(1, 1, 1)$ and $(1, x, x^2)$ form a basis of a two-dimensional subspace in $\mathbb{C}^3$?
4. Under what conditions on the scalar x do the vectors $(0, 1, x)$, $(x, 0, 1)$, and $(x, 1, 1 + x)$ form a basis $\mathbb{C}^3$?

Problem 11 1. Which of the following three definitions of transformations on $\mathbb{R}^2$ give linear transformations? (The equations are intended to hold for arbitrary real scalars $\alpha, \beta, \gamma, \delta$)

$$\begin{aligned} T(x,y) &= (\alpha x + \beta y, \gamma x + \delta y) \\ T(x,y) &= (\alpha x^2 + \beta y^2, \gamma x^2 + \delta y^2) \\ T(x,y) &= (\alpha^2 x + \beta^2 y, \gamma^2 x + \delta^2 y) \end{aligned} \tag{1}$$

2. Which of the following three definitions of transformations on the space of polynomials P give linear transformations? (The equations are intended to hold for arbitrary polynomials p.)

$$\begin{aligned} Tp(x) &= p(x^2) \\ Tp(x) &= (p(x))^2 \\ Tp(x) &= x^2 p(x) \end{aligned} \tag{2}$$

Problem 12 What are the null-spaces of the linear transformations named below?

1. The linear transformation T defined by integration:

$$Tp(x) = \int_{-3}^{x+9} p(t)dt,$$

   from P_6 to P_7.

2. The linear transformation D of differentiation on P_5.

3. The linear transformation T on $\mathbb{R}^2$ defined by

$$T(x, y) = (2x + 3y, 7x - 5y)$$

4. The transformation T from P_5 to P_{20} defined by the change of variables

$$Tp(x) = p(x^4);$$

5. The linear transformation T on $\mathbb{R}^2$ defined by

$$T(x, y) = (x, 0).$$

6. The linear transformation F from $\mathbb{R}^6$ to $\mathbb{R}^1$ defined by

$$F(x_1, \ldots, x_6) = \sum_{i=1}^{6} x_i.$$

Construct bases of the null spaces and extend them to bases of the ambient space.

Problem 13 1. If S is a linear transformation on $\mathbb{R}[x]$ defined by

$$S\,p(x) = p(x^2),$$

and T is the multiplication transformation defined by

$$T\,p(x) = x^2 p(x),$$

do S and T commute?

2. If S isa linear transformation on $P_3(\mathbb{R})$ by

$$S\,p(x) = p(x + 2),$$

and T is the transformation defined by

$$T(\alpha + \beta x + \gamma x^2 + \delta x^3) = \alpha + \gamma x^2,$$

(for all $\alpha, \beta, \gamma, \delta$) do S and T commute?

Problem 14 1. Is the linear transformation defined by

$$T(x, y) = (2y + x, 2y + x)$$

invertible?

2. What about

$$T(x, y) = (y, x)$$

3. Is the differentiation transformation D on the vector space P_5 invertible?

Problem 15 A linear map $T : P_3 \to \mathbb{R}^2$ is defined by the formula $T(f) = (f(0), f(1))$. Define an isomorphism $N(T) \to \mathbb{R}^k$ for a suitable k. You have to determine k first. Extend a basis in $N(T)$ to a basis in P_3.

Problem 16 Find the matrices of a linear transformation $T(a, b) = (2x - 3y, 5x + 7y)$ in the bases $\beta = \{(1, 3), (1, 4)\}$ and $\beta' = \{(3, 2), (7, 5)\}$. Find $Q = [I]_{\beta'}^{\beta}$. Verify $Q[T]_{\beta'} = [T]_{\beta}Q$.