MAT 200: PRACTICE MIDTERM SOLUTIONS

MON, MAR 7, 2016

Your name:____
(please print)

This is a practice midterm. It is longer than the actual one — to give you more practice.
No books or notes other than the handout posted on course web page.
Notation:
$\mathbb{Z}$ — integer numbers
$\mathbb{Z}_+$ — positive integers
$\mathbb{R}$ — real numbers

1. A (fictional) tax form contains the following statement:

You can use the tax deduction if you do not itemize deductions and your adjusted gross income is less than $54,000(61,\ 000$ for married filing jointly).

Write this statement in the language of propositional logic using notation

D: you can use the tax deduction

I: you itemize deductions

M: your filing status is "married filing jointly"

P: your adjusted gross income is less than \$54, 000

Q: your adjusted gross income is less than \$61, 000

Solution: This can be written as

$$\Big(\neg I \land (\text{your income is lower than the cutoff})\Big) \implies D$$

The income condition can be written as $P \lor (M \land Q)$. Thus, the answer is

$$\Big(\neg I \land (P \lor (M \land Q))\Big) \implies D$$

Note: the income condition can be also written as $(\neg M \land P) \lor (M \land Q)$. Both forms are correct (in fact, they are equivalent.)

2. Consider the following statement:

"Any student in this class who gets an A for the midterm is either a genius or a hard worker".

(a) Rewrite this statement using quantifiers, logical connectives, and the following notation:

S — set of all students in the class
$A(x)$: student x gets an A for the midterm.
$G(x)$: student x is a genius
$H(x)$: student x is a hard worker

Solution:

$$\forall x \in S : A(x) \implies (G(x) \vee H(x))$$

(b) Write the negation of this statement, both in formal language using quantifiers and in plain English.

Solution:

$$\neg\Big(\forall x \in S : A(x) \implies (G(x) \vee H(x))\Big)$$

Using the laws of logic, we can rewrite this as

$$\exists x \in S : \neg(A(x) \implies (G(x) \vee H(x)))$$

or, equivalently

$$\exists x \in S : (A(x) \wedge \neg(G(x) \vee H(x))$$

In plain English: some students in the class get A for the midterm without being geniuses or working hard.

3. Let the sequence a_k be defined by $a_1 = 1$, $a_{n+1} = 3a_n + 4$. Guess a general formula for a_n and prove it using induction. (Hint: compare the sequence a_n with the sequence 3, 9, 27, 81,...)

Solution: The first several terms are:

$$a_1 = 1, \ a_2 = 7, \ a_3 = 25, \ a_4 = 79$$

Comparing it with 3, 9, 27, 81, we can make the guess that

$$a_n = 3^n - 2 \tag{1}$$

Now, let us prove it by induction

- **Induction base.** For $n = 1$, formula (1) becomes $a_1 = 3 - 2$, which is true.
- **Induction step.** Assume that for some n we have $a_n = 3^n - 2$. We need to show that then $a_{n+1} = 3^{n+1} - 2$.
  Indeed,

$$\begin{aligned} a_{n+1} &= 3a_n + 4 = 3(3^n - 2) + 4 \qquad \text{(by induction assumption)} \\ &= 3^{n+1} - 6 + 4 = 3^{n+1} - 2 \end{aligned}$$

  as desired.

Thus, by induction principle, formula (1) is true for all $n \geq 1$.

4. Prove the following statement

$$\forall x \in \mathbb{R} - \{0\}, \ x + \tfrac{1}{x} > 0 \implies x > 0$$

You can use all the usual properties of real numbers, including properties of inequalities.

Solution: Let $x \in \mathbb{R} - \{0\}$. We need to prove that then

$$x + \tfrac{1}{x} > 0 \implies x > 0 \tag{2}$$

Since $x \neq 0$, either $x > 0$ or $x < 0$. Consider these two cases separately.

Case 1: $x > 0$. Then $x + \frac{1}{x} > 0$ is true, and $x > 0$ is also true, so (2) is true.

Case 2: $x < 0$. Then $\frac{1}{x} < 0$, so $x + \frac{1}{x} < 0$. Thus, in this case (2) becomes $F \implies F$, which is true. So, in this case (2) is also true.

5. For each of the following statements tell whether it is true or false, and justify your answer by giving a proof. You can use all the usual properties of real numbers, including properties of inequalities.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : 0 < y < x$ TRUE FALSE

Solution: FALSE. Counterexample: take $x = -1$; then there is no y satisfying $0 < y < -1$.

(b) $\forall x \in \mathbb{R}_+, \exists y \in \mathbb{R}_+ : 0 < y < x$ TRUE FALSE
(here $\mathbb{R}_+$ is the set of positive real numbers)

Solution: TRUE. For any $x > 0$, we can take $y = x/2$; then $0 < y < x$ is true.

(c) $\exists y \in \mathbb{R}_+, \forall x \in \mathbb{R}_+ : 0 < y < x$ TRUE FALSE
(here $\mathbb{R}_+$ is the set of positive real numbers)

Solution: FALSE. This is saying that there is a largest positive real number. This is false: for any $y \in \mathbb{R}_+$ we can find an $x \in \mathbb{R}_+$ such that $x > y$ — for example, take $x = y + 1$.

6. Prove that for any three sets A, B, C we have $A - (B \cup C) = (A - B) \cap (A - C)$. Note: you can use Venn diagrams to illustrate your proof; however, Venn diagrams by themselves will not be considered sufficient proof.

Solution: We need to prove two implications:

$$\Big(x \in A - (B \cup C)\Big) \implies \Big(x \in (A - B) \cap (A - C)\Big)$$

$$\Big(x \in (A - B) \cap (A - C)\Big) \implies \Big(x \in A - (B \cup C)\Big)$$

Asume $x \in A - (B \cup C)$. Then $x \in A$, $x \notin (B \cup C)$. The last condition is equivalent to $x \notin B \wedge x \notin C$. Thus, $x \in A - B$. Similarly, $x \in A - C$. Therefore, $x \in (A - B) \cap (A - C)$.

Conversely, assume $x \in x \in (A - B) \cap (A - C)$. Then $x \in A - B$ and $x \in A - C$. Thus, $x \in A$ and $x \notin B$; similarly $x \notin C$. Since $x \notin B$, $x \notin C$, we get $x \notin B \cup C$. Thus, $x \in A - (B \cup C)$.