(2) Average value:

$$\frac{1}{b}\int_0^b (x^2+x)\,dx = \frac{1}{b}\left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^b$$

$$= \frac{1}{b}\cdot\left(\frac{1}{3}b^3+\frac{1}{2}b^2\right)$$

$$= \frac{1}{3}b^2+\frac{1}{2}b$$

$$\frac{1}{3}b^2+\frac{1}{2}b=4$$

$$2b^2+3b=24$$

$$2b^2+3b-24=0$$

$$b=\frac{-3\pm\sqrt{3^2+4\cdot24\cdot2}}{4} = \frac{-3\pm\sqrt{201}}{4}$$

Since we want b to be positive,

$$b=\frac{-3+\sqrt{201}}{4}$$

③ ⓐ $y' = 2y - e^x$

$x_0 = 0 \quad y_0 = 1 \quad y' = 2 - 1 = 1$

$x_1 = 0.5 \quad y_1 = 1 + 1 \cdot 0.5 = 1.5 \qquad y' = 2 \cdot 1.5 - e^{0.5} = 3 - \sqrt{e}$

$x_2 = 1 \quad y_2 = 1.5 + 0.5 \cdot (3 - \sqrt{e}) = \boxed{3 - \frac{\sqrt{e}}{2}}$

ⓑ $y' = \sin(x) \cdot y + 1$

$x_0 = 1 \quad y_0 = 2 \quad y' = \sin(1) \cdot 2 + 1 \approx 2.68$

$x_1 = 1.2 \quad y_1 = 2 + 0.2 \cdot 2.68 \approx 2.54 \qquad y_1' = \sin(1.2) \cdot 2.54 + 1 \approx 3.37$

$x_2 = 1.4 \quad y_2 = 2.54 + 0.2 \cdot 3.37 \approx 3.21$

Note: this problem would be very messy if we were required to write exact answers. In real exam, things will be simpler

(4)

$$x^3 y'' - 2x^2 y' = -1$$

(a) $y = c_1 x^3 + c_2 - \frac{1}{4x}$

$$y' = 3c_1 x^2 + \frac{1}{4x^2}$$

$$y'' = 6c_1 x - \frac{2}{4x^3} = 6 c_1 x - \frac{1}{2x^3}$$

$$x^3 y'' - 2x^2 y' =$$

$$= \left(6c_1 x^4 - \frac{1}{2}\right) - 2\left(3c_1 x^4 + \frac{1}{4}\right)$$

$$= 6c_1 x^4 - \frac{1}{2} - 6c_1 x^4 - \frac{1}{2}$$

$$= -1$$

as required

(b) $c_1 = 1 \Rightarrow \quad y = x^3 + c_2 - \frac{1}{4x}$

$$y(0.5) = \left(\tfrac{1}{2}\right)^3 + c_2 - \frac{1}{4\cdot \frac{1}{2}}$$

$$= \frac{1}{8} + c_2 - \frac{1}{2}$$

$$= -\frac{3}{8} + c_2$$

$y(0.5) = 1 \Rightarrow \quad -\frac{3}{8} + c_2 = 1$

$$c_2 = 1\tfrac{3}{8} = \frac{11}{8}$$

$$y = x^3 + \frac{11}{8} - \frac{1}{4x}$$

(c) $y(1)=0 \Rightarrow c_1 \cdot 1^3 + c_2 - \frac{1}{4} = 0$

$y(2)=1 \Rightarrow c_1 \cdot 8 + c_2 - \frac{1}{8} = 1$

$$\begin{cases} c_1 + c_2 = \frac{1}{4} \\ 8c_1 + c_2 = \frac{9}{8} \end{cases}$$

$$\begin{cases} c_2 = \frac{1}{4} - c_1 \\ 8c_1 + \left(\frac{1}{4} - c_1\right) = \frac{9}{8} \end{cases}$$

$7c_1 + \frac{1}{4} = \frac{9}{8}$

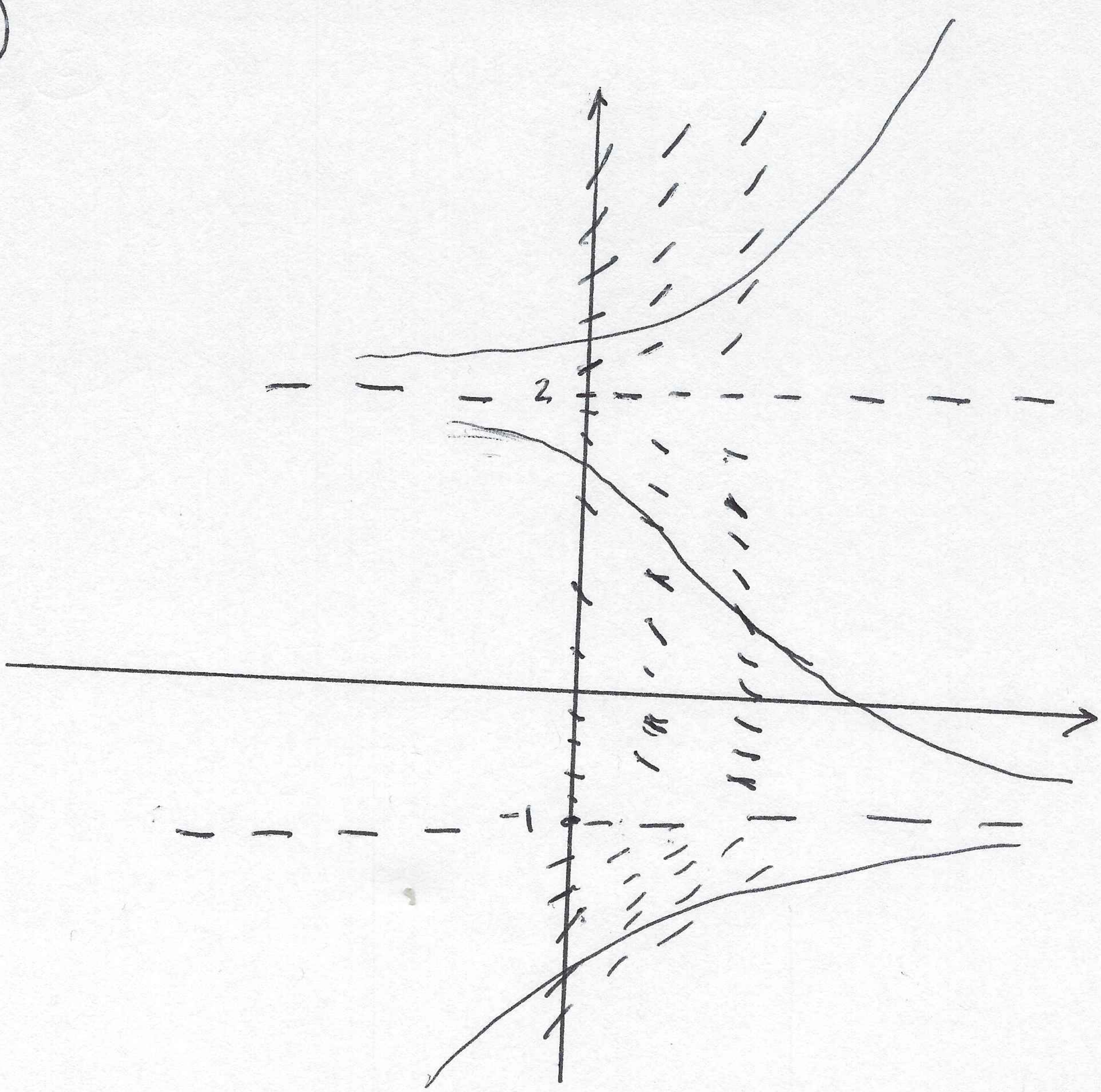
$7c_1 = \frac{9}{8} - \frac{1}{4} = \frac{7}{8}$

$c_1 = \frac{1}{8}$

$c_2 = \frac{1}{8}$

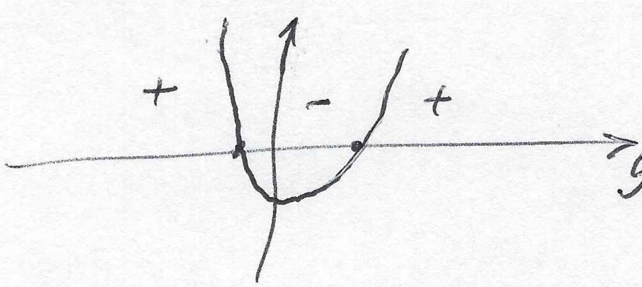
$y = \frac{1}{8}x^3 + \frac{1}{8} - \frac{1}{4x}$

5



$y^2 - y - 2 = (y-2)(y+1)$

Graph of $(y-2)(y+1)$:

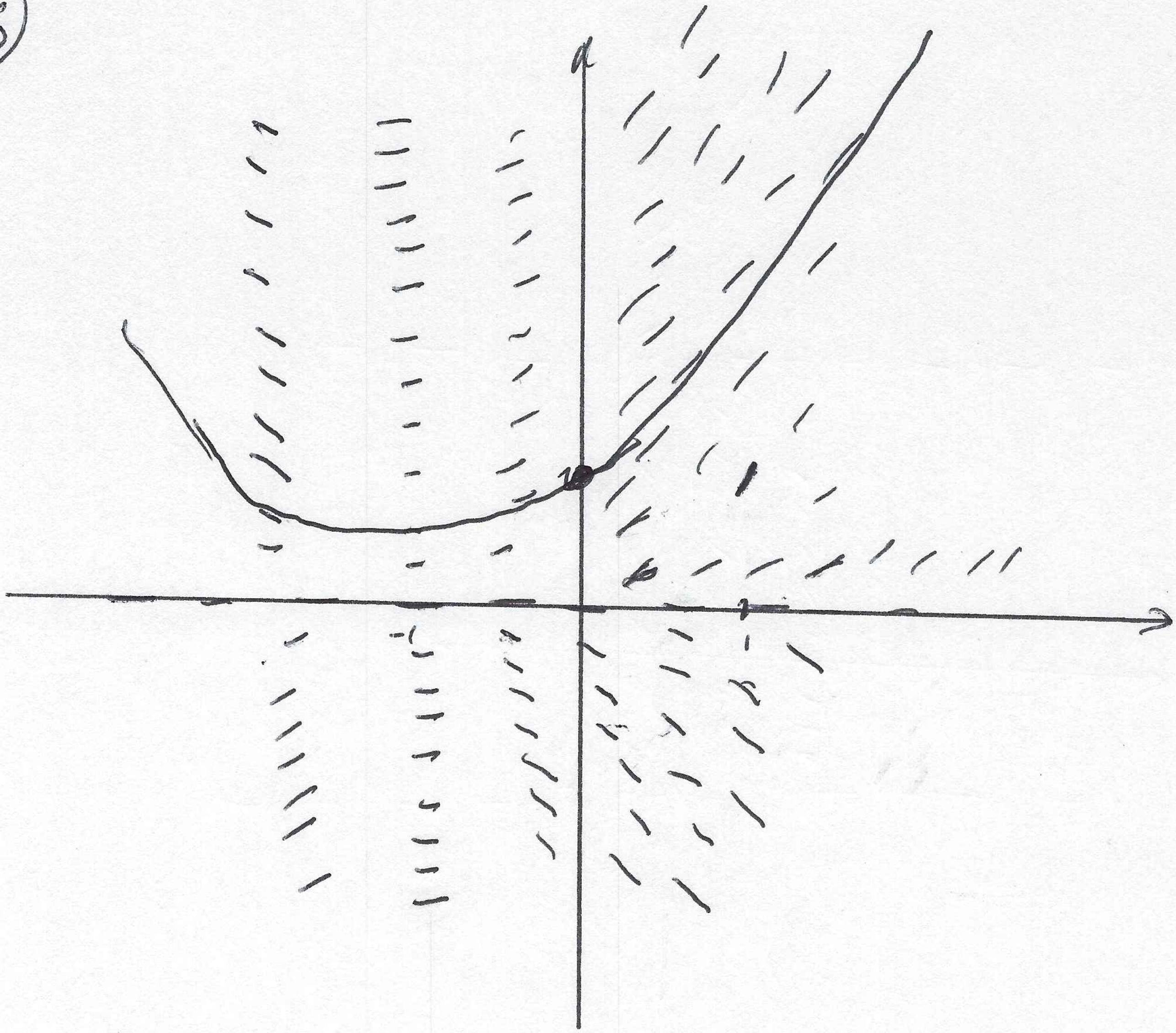


for $y(0) \leq -1$, solution $\to -1$ as $x \to \infty$

for $-1 \leq y(0) < 2$, solution $\to -1$

for $y(0) > 2$, solution $\to +\infty$

⑥



$y' = y + xy = y(x+1)$

7

$$xy^2 - y'x^2 = 0$$

$$y'x^2 = xy^2$$

$$\frac{y'}{y^2} = \frac{x}{x^2} = \frac{1}{x}$$

$$\int \frac{dy}{y^2} = \int \frac{dx}{x}$$

$$-\frac{1}{y} = \ln|x| + C$$

$$y = \frac{-1}{\ln|x| + C}$$

(8)

$$yy' = (y+1)\ln(x)$$

$$\frac{y}{y+1}\,y' = \ln(x)$$

$$\int \frac{y\,dy}{y+1} = \int \ln(x)\,dx$$

Left-hand side:

$$\int \frac{y\,dy}{y+1} = \int \frac{(u-1)}{u}\,du = \int 1 - \frac{1}{u}\,du$$

$$u = y+1$$

$$= u - \ln|u| + C_1 = y+1 - \ln|y+1| + C_1$$

RHS:

$$\int \ln(x)\,dx = x\ln(x) - x + C_2$$

(computed in class)

$$\boxed{y - \ln|y+1| = x\ln(x) - x + C}$$

9 Original family

$$y = x^3 + C$$

Diff. eqn:

$$y' = 3x^2$$

Orth. family: $y' = -\frac{1}{3x^2}$

$$dy = -\frac{dx}{3x^2}$$

$$\int dy = -\int \frac{dx}{3x^2}$$

$$y = \frac{1}{3} \cancel{2} \cdot \frac{1}{x} + C$$

$$y = \frac{1}{3x} + C$$

(10) (a) Let $P(t)$ be the amount of polluted water in the aquarium at time t (water is measured in gallons, time in hours).

Then:

$$\frac{dP}{dt} = -5 \cdot \frac{P(t)}{10}$$

$\uparrow$ drain rate

$\frac{P(t)}{10}$: concentration of polluted water

$$\frac{dP}{dt} = -\frac{P(t)}{2} = -\frac{1}{2}P(t)$$

Solution: $P(t) = A \cdot e^{-\frac{1}{2}t}$ (discussed in class)

To find A: $P(0) = A \cdot e^{0} = 10$

$\Downarrow$

$A = 10$

$$P(t) = 10 \cdot e^{-t/2}$$

To find when half polluted water is gone:

$$P(t) = 10 \cdot e^{-t/2} = \frac{10}{2} = 5$$

$$2e^{-t/2} = 1$$

$$e^{-t/2} = \frac{1}{2}$$

$$-\frac{t}{2} = \ln\left(\frac{1}{2}\right)$$

$$t = -2\ln\left(\frac{1}{2}\right) = 2\ln(2)$$

(b) $\frac{dP}{dt} = k \cdot P \Rightarrow P = A \cdot e^{kt}$

(t = time since 1990, in years)
P – in millions

$P(0) = 30 \Rightarrow A \cdot e^0 = 30$
$A = 30$

$P(t) = 30 \cdot e^{kt}$

To find k: at $t = 10$ (i.e., year 2000)

$P(10) = 30 \cdot e^{k \cdot 10} = 34$

$e^{10k} = \frac{34}{30}$

$10k = \ln\left(\frac{34}{30}\right)$

$k = \frac{1}{10} \cdot \ln\left(\frac{34}{30}\right)$

Then: $P(30) = 30 \cdot e^{k \cdot 30}$

$= 30 \cdot e^{3 \ln\left(\frac{34}{30}\right)}$

$= 30 \cdot \left(\frac{34}{30}\right)^3$

(11) Let $r(t)$ = fraction of people who have heard the rumor

(a) $\dfrac{dr}{dt} = k \cdot r \cdot (1-r)$ — logistic eqn

(b) $r = \dfrac{1}{1 + A \cdot e^{-kt}}$ (proved in the textbook)

(c) Let t = time since 8 am

At $t=0$, $r = \dfrac{80}{1000} = 0.08$

At $t=4$, $r = 0.5$

$$\begin{cases} \dfrac{1}{1+A\cdot e^{-k\cdot 0}} = 0.08 \\[2ex] \dfrac{1}{1+A\cdot e^{-k\cdot 4}} = 0.5 \end{cases}$$

$$\frac{1}{0.08} = 1 + A \Rightarrow A = \frac{1}{0.08} - 1$$

$$= \frac{100}{8} - 1 = 12.5 - 1 = 11.5$$

so, $$\frac{1}{1+11.5\cdot e^{-4k}} = 0.5$$

$$1+11.5\cdot e^{-4k} = 2$$

$$11.5\cdot e^{-4k} = 1$$

$$e^{-4k} = \frac{1}{11.5} = \frac{2}{23}$$

$$-4k = \ln\left(\frac{2}{23}\right)$$

$$k = -\frac{\ln\left(\frac{2}{23}\right)}{4} = \frac{\ln\left(23/2\right)}{4}$$

To find when 80% will hear it:

$$\frac{1}{1+11.5\cdot e^{-kt}} = 0.8, \quad \text{where } k \text{ is above}$$

$$1+11.5\cdot e^{-kt} = \frac{1}{0.8} = \frac{10}{8} = 1.25$$

$$11.5\, e^{-kt} = 0.25$$

$$e^{-kt} = \frac{0.25}{11.5} = \frac{1}{46}$$

$$-kt = \ln(1/46) \qquad kt = \ln(46)$$

$$t = \ln(46)/k = \ln(46)/(\ln(23/2)/4) = \frac{4\ln(46)}{\ln\left(\frac{23}{2}\right)}$$