# Midterm 2 Solutions

MAT 127

Nov 2, 2015

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|---|---|
| **Name**: (please print) | **ID #**: |
| **Your section:** (see list below) | |

| | 1 | 2 | 3 | 4 | 5 | **Total** |
|---|---|---|---|---|---|---|
| | 20pt | 20pt | 20pt | 10pt | 20pt | 90pts |
| *Grade* | | | | | | |

**No notes, books or calculators.**

**You must show your reasoning, not just the answer. Answers without justification will get only partial credit.**

**Please cross out anything that is not part of your solution — e.g., some preliminary computations that you didn't need.**

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|---|---|---|
| Lecture 01 | MWF 10:00 AM – 10:53 AM | Alexander Kirillov |
| Lecture 02 | MW 5:30 PM – 6:50 PM | Mark McLean |
| Lecture 04 | TUTH 5:30 PM – 6:50 PM | Sabyasachi Mukherjee |

**1.** *(20 pts)*

(a) Calculate the degree 2 Taylor polynomial $T_2(x)$ of $xe^{2x}$ centered at $a = 1$.

**Answer:**
We have that $\frac{d}{dx}(xe^{2x}) = e^{2x}+2xe^{2x} = (1+2x)e^{2x}$ and $\frac{d^2}{dx^2}(xe^{2x}) = 2e^{2x}+2(1+2x)e^{2x} = (4+4x)e^{2x}$. Then $T_2(x) = e^2 + 3e^2(x-1) + 4e^2(x-1)^2$.

(b) Show that $|xe^{2x} - T_2(x)| \leq 4e^3|x-1|^3$ in the interval $0.5 \leq x \leq 1.5$.

**Answer:**
We use Taylor's Inequality. So if $R_2(x) = xe^{2x} - T_2(x)$ then $|R_2(x)| \leq \frac{M}{3!}|x-1|^3$ in the interval $0.5 \leq x \leq 1.5$. Here $M$ is the maximum of $\frac{d^3}{dx^3}(xe^{2x})$ in the interval $0.5 \leq x \leq 1.5$. We have that $\frac{d^3}{dx^3}(xe^{2x}) = (12+8x)e^{2x}$. To maximize this function we need to find its critical points inside $0.5 \leq x \leq 1.5$. Hence we need to calculate $\frac{d}{dx}(12+8x)e^{2x} = (32+16x)e^{2x}$. This is never zero for $0.5 \leq x \leq 1.5$ and hence its maximum is on the endpoints $x = 1.5$ or $x = 0.5$. We have $(12 + 8 \times 0.5)e^{2\times 0.5} = 16e$ and $(12+8\times 1.5)e^{2\times 1.5} = 24e^3$. Hence $M = 24e^3$. Hence $|xe^{2x} - T_2(x)| \leq \frac{24e^3}{6}|x-1|^3 = 4e^3|x-1|^3$.

Another way to show that $M = 24e^3$: we can notice that $(12+8x)e^{2x}$ is an increasing function, so the maximum is achieved at $x = 1.5$.

**2.** *(20 pts)*

(a) For which $c$ is $y = xe^{cx}$ a solution of

$$y'' - 4y' + 4y = 0.$$

**Answer:**
$y' = (1+cx)e^{cx}$ and $y'' = (2c+c^2x)e^{cx}$. So: $y''-4y'+4 = (2c+c^2x)e^{cx} - 4(1+cx)e^{2x} + 4xe^{cx} = (2c-4+(c^2-4c+4)x)e^{cx} = 0$. So $2c-4=0$ and so $c=2$.

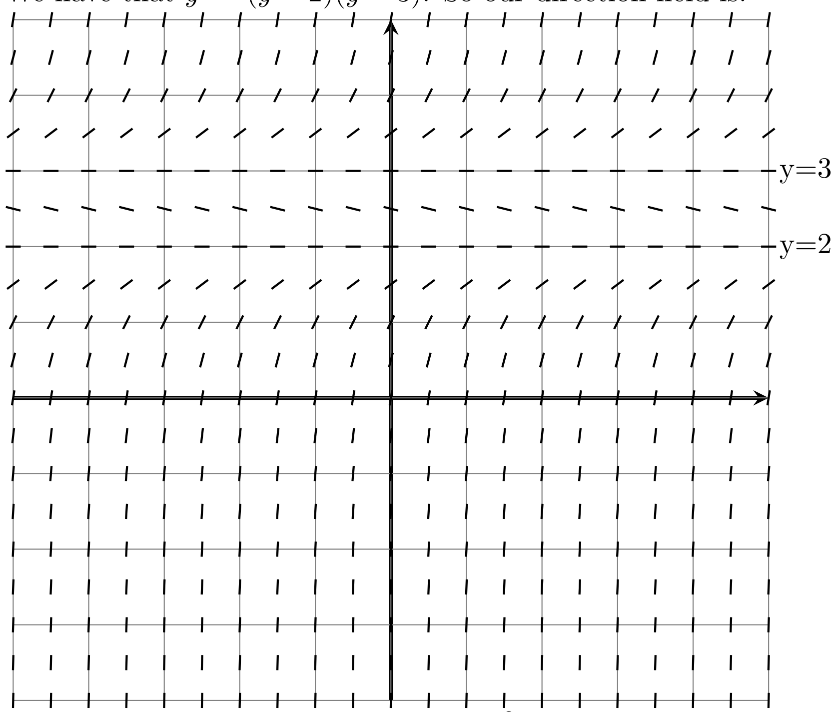
(b) What are the solutions of the equation $y'' = y^2 - 2yx + x^2$ of the form $y = ax + b$ where $a$ and $b$ are constants?

**Solution:** If $y = ax + b$ is a solution then $y'' = 0$ and hence $y^2 - 2yx + x^2 = 0$. Hence $(ax+b)^2 - 2x(ax+b) + x^2 = a^2x^2 + 2abx + b^2 - 2ax^2 - 2bx + x^2 = (a^2 - 2a + 1)x^2 + (2ab - 2b)x + b^2 = 0$. Hence $b = 0$ and $a^2 - 2a + 1 = 0$. Therefore $a = 1$. Hence $y = x$ is the only such solution.

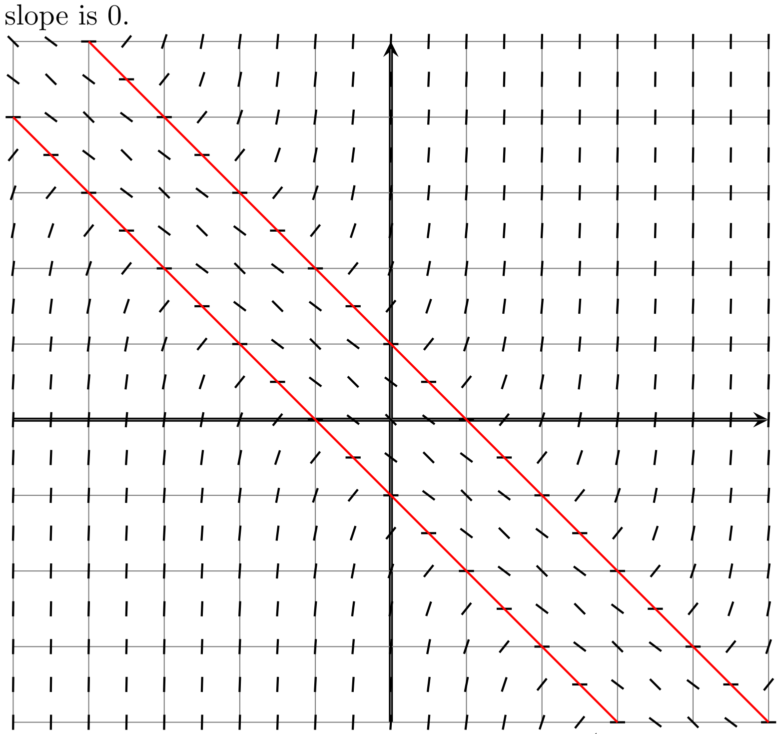
**3.** *(20 pts)*

(a) Draw the direction field for $y' = y^2 - 5y + 6$. Make sure you draw the region where the slope is 0.

**Answer:**
We have that $y' = (y-2)(y-3)$. So our direction field is:




(b) Draw the direction field for $y' = (x+y)^2 - 1$. Make sure you draw the region where the slope is 0.



$y = -x - 1$, $y = -x + 1$ (where slope is 0)

**4.** *(10 pts)*

Use Eulers method with step size 0.1 to estimate $y(0.3)$ where $y(x)$ satisfies:

$$y' = x + y, \quad y(0) = 1.$$

**Solution:**

So $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$. We have: $y_0 = 1$, $y_1 = 1 + (0 + 1) \times 0.1 = 1.1$, $y_2 = 1.1 + (0.1 + 1.1) \times 0.1 = 1.22$, $y_3 = 1.22 + (0.2 + 1.22) \times 0.1 = 1.362$.

**5.** *(20 pts)*
Solve:

$$(yx^2 - y)y' = 1, \quad y(0) = 1.$$

**Answer:**
We have:

$$yy' = \frac{1}{x^2 - 1}$$

and so

$$\int y dy = \frac{y^2}{2} = \int \frac{1}{x^2-1} dx = \int \frac{1}{(x-1)(x+1)} dx$$

We have that:

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

and so:

$$1 = A(x+1) + B(x-1) = (A+B)x + A - B$$

and so $A+B=0$ and $A-B=1$. Hence $A=-B$ and so $2A=1$ and so $A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

Therefore $\int \frac{1}{(x-1)(x+1)} dx = \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)} dx = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C$. Hence $\frac{y^2}{2} = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C$. And so $y = \sqrt{(\ln|x-1| - \ln|x+1|) + C/2}$ or $y = -\sqrt{(\ln|x-1| - \ln|x+1|) + C/2}$. Because $y(0) = \sqrt{C/2} = 1$, we get that $C = 2$. Hence $y = \sqrt{(\ln|x-1| - \ln|x+1|) + 1}$.