MAT 127, MIDTERM 1
PRACTICE PROBLEMS

The midterm covers chapters 8.1 — 8.6 in the textbook. The actual exam will contain 5 problems (some multipart), so it will be shorter than this practice exam.

1. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = (-1)^n \frac{n+4}{n^3 - 2n^2 + 4}$$

Solution:

$$\lim_{n\to\infty} (-1)^n \frac{n+4}{n^3-2n^2+4} = \lim_{n\to\infty} (-1)^n \frac{n+4}{n^3-2n^2+4} \times \frac{1/n^3}{1/n^3}$$
$$= \lim_{n\to\infty} (-1)^n \frac{1/n^2 + 4/n^3}{1 - 2/n + 4/n^3} = \frac{0+0}{1-0+0} = 0$$

2. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = \frac{3^n + 1}{n!}$$

Solution:

$$\lim_{n\to\infty} \frac{3^n+1}{n!} = \lim_{n\to\infty} \frac{3^n}{n!} + \frac{1}{n!} = \lim_{n\to\infty} \frac{3^n}{1.2.3.4\cdots n} + \frac{1}{n!}$$
$$= \lim_{n\to\infty} \frac{3^{n-1}}{1.2.3.4\cdots n-1} \times \frac{3}{n} + \frac{1}{n!} = \lim_{n\to\infty} \frac{3^2}{2} \times \frac{3^{n-3}}{3.4.5\cdots n-1} \times \frac{3}{n} + \frac{1}{n!}.$$

Now, $3^{n-3} = 3.3.3.3\cdots .3.3 \leq 3.4.5\cdots n-1$. Hence $\frac{3^{n-3}}{3.4.5\cdots n-1} \leq 1$. Therefore:

$$0 \leq \frac{3^2}{2} \times \frac{3^{n-3}}{3.4.5\cdots n-1} \times \frac{3}{n} + \frac{1}{n!} \leq \frac{3^2}{2} \times \frac{3}{n} + \frac{1}{n!} = \frac{27}{2n} + \frac{1}{n!}.$$

Because $\lim_{n\to\infty} \frac{27}{2n} + \frac{1}{n!} = 0$, we have by the squeeze theorem that

$$\lim_{n\to\infty} \frac{3^{n-3}}{3.4.5\cdots n-1} \times \frac{3}{n} + \frac{1}{n!} = 0$$

and hence

$$\lim_{n\to\infty} \frac{3^n+1}{n!} = 0.$$

3. Determine whether the following sequence converges. If it converges, find the limit

$$a_n = \frac{(\ln n)^2}{n}$$

Solution:

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{\frac{1}{x} 2\ln(x)}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{2}{x}\right)}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

by applying L'Hôpitals rule twice.

4. Let the sequence a_n be defined by $a_1 = 1$, $a_{n+1} = \dfrac{3 + a_n}{2}$ for $n \geq 1$.

(a) Show that this sequence is bounded: $a_n \leq 3$ for all n.

Solution:
First of all, $a_1 = 1 < 3$.
Now suppose that $a_n < 3$. We wish to show that $a_{n+1} < 3$ assuming that the previous term a_n satisfies $a_n < 3$. We have that $a_{n+1} = \frac{3+a_n}{2} = \frac{3}{2} + \frac{a_n}{2}$. Since we have assumed that $a_n < 3$, we get that $\frac{a_n}{2} < \frac{3}{2}$ and hence $\frac{3}{2} + \frac{a_n}{2} < \frac{3}{2} + \frac{3}{2} = 3$. Hence $a_{n+1} = \frac{3}{2} + \frac{a_n}{2} < 3$.
Therefore we have shown that $a_n < 3$ for all n.

(b) Explain why this sequence is convergent and find the limit.

Solution:
The sequence a_n is increasing for the following reason: We have that $a_{n+1} = \frac{3+a_n}{2} = \frac{3}{2} + \frac{a_n}{2}$. Now since $a_n < 3$, we get that $\frac{3}{2} > \frac{a_n}{2}$ and hence $\frac{3}{2} + \frac{a_n}{2} > \frac{a_n}{2} + \frac{a_n}{2} = a_n$. Hence $a_{n+1} = \frac{3}{2} + \frac{a_n}{2} > a_n$. Therefore a_n is an increasing sequence. Hence a_n converges by the *monotone convergence theorem*.
We now need to find its limit. Suppose that $\lim_{n\to\infty} a_n = L$. Replacing n with $n+1$ in the above limit, we get that $\lim_{n+1\to\infty} a_{n+1} = L$ and hence $\lim_{n\to\infty} a_{n+1} = L$. We have the equation:

$$a_{n+1} = \frac{3 + a_n}{2}.$$

Taking the limits of both sides as $n \to \infty$ gives us:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3 + a_n}{2}$$

and so:

$$L = \frac{3 + L}{2}$$

and so $L = 3$ by solving the above equation. Hence $\lim_{n\to\infty} a_n = 3$.

5. If $\displaystyle\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms, what can you say about the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \sin(a_n)$? Does it converge? Does it converge absolutely?

Solution:
Since $\sum_{n=1}^{\infty} a_n$ is convergent, we get that $\lim_{n\to\infty} a_n = 0$. Hence $\lim_{n\to\infty} \frac{\sin a_n}{a_n} = \lim_{x\to 0} \frac{\sin x}{x} = 1$. Therefore by the limit comparison test, $\sum_{n=1}^{\infty} \sin(a_n)$ converges.
It is also absolutely convergent since $0 < a_n < \pi$ for all n sufficiently large and hence $\sin(a_n)$ is positive for all n sufficiently large. Here we used the fact that absolute convergence is equivalent to convergence in the case when series have positive terms for all sufficiently large n.

6. For which values of p is the series $\displaystyle\sum_{n=1}^{\infty} p^n \frac{n!}{(2n)!}$ convergent?

Solution:
Because this is a power series in p, we use the ratio test.

$$\lim_{n\to\infty}\left|\frac{p^{n+1}\frac{n+1!}{(2(n+1))!}}{p^n\frac{n!}{(2n)!}}\right| = \lim_{n\to\infty}|p|\frac{n+1}{(2n+1)(2(n+1))} = \lim_{n\to\infty}|p|\frac{n+1}{4n^2+6n+2} = 0.$$

Hence the radius of convergence is ∞. Hence this power series converges for all p.

7. If the series $\displaystyle\sum_{n=1}^{\infty} c_n 4^n$ is divergent, what can you say about the following series:

$$(a)\sum_{n=1}^{\infty} c_n 2^n, \qquad (b)\sum_{n=1}^{\infty} c_n(-8)^n, \qquad (c)\sum_{n=1}^{\infty} c_n(-4)^n.$$

Solution:

(a) The series $\sum_{n=1}^{\infty} c_n 2^n$ may or may not converge. For instance if $c_n = 1$ then both $\sum_{n=1}^{\infty} c_n 2^n$ and $\sum_{n=1}^{\infty} c_n 4^n$ diverge. But, if $c_n = \frac{1}{4^n}$ then $\sum_{n=1}^{\infty} c_n 2^n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges as it is a geometric series, but $\sum_{n=1}^{\infty} c_n 4^n = \sum_{n=1}^{\infty} 1$ diverges.

(b) The series $\sum_{n=1}^{\infty} c_n(-8)^n$ must diverge.
This is true for the following reason: Since $8^n = 2^n 4^n$,

$$\lim_{n\to\infty}|c_n|8^n = \lim_{n\to\infty}|c_n 4^n|2^n = \lim_{n\to\infty}\frac{|c_n 4^n|}{\frac{1}{2^n}}.$$

Now $\lim_{n\to\infty}\frac{|c_n 4^n|}{\frac{1}{2^n}}$ cannot be 0 since this would imply that $\sum_{n=1}^{\infty} c_n 4^n$ is convergent by the limit comparison test. Hence $\lim_{n\to\infty}|c_n|8^n$ is non-zero or non-convergent and so $\sum_{n=1}^{\infty} c_n(-8)^n$ must diverge by the alternating series test.

(c) $\sum_{n=1}^{\infty} c_n(-4)^n$ may or may not converge.
For instance if $c_n = 1$, then $\sum_{n=1}^{\infty} c_n 4^n = \sum_{n=1}^{\infty} 4^n$ diverges and also $\sum_{n=1}^{\infty} c_n(-4)^n$ diverges by the alternating series test since $\lim_{n\to\infty} 4^n \neq 0$.
On the other hand, if $c_n = \frac{1}{n4^n}$ then $\sum_{n=1}^{\infty} c_n 4^n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} c_n(-4)^n$ converges by the alternating series test since $\lim_{n\to\infty} c_n 4^n = \lim_{n\to\infty} \frac{1}{n4^n}4^n = \lim_{n\to\infty}\frac{1}{n} = 0$.

8. Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{1+(-2)^n}{3^n}$$

Solution:

$$\sum_{n=0}^{\infty} \frac{1+(-2)^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} + \frac{(-2)^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} + \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = \frac{1}{1-\frac{1}{3}} + \frac{1}{1-(-\frac{2}{3})} = \frac{21}{10}.$$

9. Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)}$$

Solution:

$$\frac{1}{(n+1)(n+3)} = \frac{A}{n+1} + \frac{B}{n+3}$$

So:

$$1 = A(n+3) + B(n+1) = (A+B)n + 3A + B.$$

Hence, $3A + B = 1$ and $A + B = 0$. Therefore $2A = 1$, and so $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Hence:

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = \sum_{n=0}^{\infty} \frac{1}{2(n+1)} - \frac{1}{2(n+3)}$$

Now:

$$s_m = \sum_{n=0}^{m} \frac{1}{2(n+1)} - \sum_{n=0}^{m} \frac{1}{2(n+3)} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2((m-1)+3)} - \frac{1}{2(m+3)} =$$

$$= \sum_{n=0}^{m} \frac{3}{4} - \frac{1}{2m+4} - \frac{1}{2m+6}.$$

So:

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+3)} = \lim_{m\to\infty} s_m = \frac{3}{4}.$$

10. Write the number $1.1\overline{009} = 1.1009009009\ldots$ as a fraction

Solution:

We will first write $0.\overline{009}$ as a fraction:

$$0.\overline{009} = \sum_{n=1}^{\infty} \frac{9}{1000^n} = \frac{\frac{9}{1000}}{1-\frac{1}{1000}} = \frac{9}{999} = \frac{1}{111}.$$

Hence:

$$1.1\overline{009} = 1.1 + \frac{1}{10} \times 0.\overline{009} = \frac{11}{10} + \frac{1}{1110} = \frac{1222}{1110} = \frac{611}{555}.$$

11. Determine whether the following series converges or diverges

$$\sum_{n=0}^{\infty} \frac{\sin(3\pi n/7)}{n^2+1}$$

Solution:
Since $\sin(x) \leq 1$ for all x,

$$\left|\frac{\sin(3\pi n/7)}{n^2+1}\right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$$

Because $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges, we get that $\sum_{n=0}^{\infty} \left|\frac{\sin(3\pi n/7)}{n^2+1}\right|$ converges by the comparison test. Hence $\sum_{n=0}^{\infty} \frac{\sin(3\pi n/7)}{n^2+1}$ is absolutely convergent and therefore convergent.

12. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n \cdot 3^n}$$

Find the radius of convergence and the interval of convergence.

Solution:
To find the radius of convergence we use the ratio test:

$$\lim_{n\to\infty} \frac{\left(\frac{(2x-1)^{n+1}}{(n+1)\cdot 3^{n+1}}\right)}{\left(\frac{(2x-1)^n}{n\cdot 3^n}\right)} = \lim_{n\to\infty}(2x-1)\frac{n3^n}{(n+1)3^{n+1}} = \lim_{n\to\infty}(2x-1)\frac{n}{3(n+1)} = \frac{(2x-1)}{3}.$$

The ratio test tells us that this converges when $\left|\frac{(2x-1)}{3}\right| < 1$ and so $-1 < \frac{2x-1}{3} < 1$ and so $-1 < x < 2$. Hence the radius of convergence is $\frac{3}{2}$.

To find the interval of convergence we only need to check the endpoints -1 and 2. If $x = -1$ then our sum is:

$$\sum_{n=1}^{\infty} \frac{(-2-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges by the alternating series test. Hence -1 is in our interval of convergence.

If $x = 2$ then our sum is:

$$\sum_{n=1}^{\infty} \frac{(4-1)^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges and hence 2 is not in our interval of convergence.

Therefore the interval of convergence is $[-1, 2)$.

13. Write the function $f(x) = \ln(1+2x)/x$ as a power series in x. Find the radius of convergence of this series.

Solution:
We will first compute the power series for $\ln(1+x)$. We have:

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = \int_0^x \frac{1}{1-(-t)}dt = \int_0^x \sum_{n=0}^{\infty}(-t)^n dt = \sum_{n=0}^{\infty}\int_0^x (-t)^n dt =$$

$$\sum_{n=0}^{\infty}\int_0^x (-1)^n t^n dt = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}x^{n+1}$$

Hence

$$\ln(1+2x)/x = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}(2x)^{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{x}\frac{2^{n+1}}{n+1}x^{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{2^{n+1}}{n+1}x^n.$$