Final exam practice problems with solutions

MAT 127

Dec 7, 2015

Name: (please print)		**ID #**:
Your section:	(see list below)	

This is a collection of practice problems for the final exam. Note that it is much longer than the actual exam.

In these solutions, we used a calculator to compute approximate decimal values of our answers. In the actual exam, you will not be required to do so: you can leave the answer as an expression involving some algebraic, trigonometric, exponential, or logarithmic functions

Lecture 01	MWF 10:00 AM – 10:53 AM	Alexander Kirillov
Lecture 02	MW 5:30 PM – 6:50 PM	Mark McLean
Lecture 04	TUTH 5:30 PM – 6:50 PM	Sabyasachi Mukherjee

1. For each of the following sequence, determine whether it converges. If it converges, find the limit.

(a) $a_n = \frac{n^3+2}{2n^4+1}$

(b) $a_n = n\sin(\pi/n)$

(c) $a_n = \sqrt{2n+1} - \sqrt{2n-1}$

(d) $a_n = \frac{2^n}{n!}$

Answer:

(a) $\lim a_n = \lim \frac{\frac{1}{n}+\frac{2}{n^4}}{2+\frac{1}{n^4}} = 0$

(b) Use L'Hopital's rule: $\lim_{x\to\infty} x\sin(\pi/x) = \lim_{t\to 0+} \frac{1}{t}\sin(\pi t) = \frac{\pi}{1}$

(c) Multiplying by $\sqrt{2n+1} + \sqrt{2n-1}$:

$$\begin{aligned} a_n &= \frac{(\sqrt{2n+1} - \sqrt{2n-1})(\sqrt{2n+1} + \sqrt{2n-1})}{\sqrt{2n+1} + \sqrt{2n-1}} \\ &= \frac{(2n+1)-(2n-1)}{\sqrt{2n+1} + \sqrt{2n-1}} = \frac{2}{\sqrt{2n+1} + \sqrt{2n-1}} \end{aligned}$$

so $\lim a_n = 0$.

(d) Since

$$\frac{a_{n+1}}{a_n} = \frac{2(n!)}{(n+1)!} = \frac{2}{n+1}$$

we see that $\lim a_{n+1}/a_n = 0$. Thus, we see that $\lim a_n = 0$.

2. Let the sequence a_n be defined by $a_0 = \pi/4$, $a_{n+1} = \sin(a_n)$. Prove that this sequence has a limit and find this limit.

Answer: It is known that for any $x \in (0, \pi)$, we have $0 < \sin(x) < x$. Thus, this sequence is monotonically decreasing and bounded below by 0. Thus, it must converge, and the limit L must satisfy $L = \sin(L)$. From this it is clear that $\lim(a_n) = 0$

3. For each of the series below, determine whether the series converges.

(a)
$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+4}$$

(b)
$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{2n+3}}$$

(c)
$$\sum_{n=0}^{\infty} \frac{n^2+4n}{n^3+1}$$

(d)
$$\sum_{n=0}^{\infty} \frac{ne^{2n}}{(2n)!}$$

Answer:

(a) Since $\lim \frac{n^2-1}{n^2+4} = 1$, this series diverges. (For a series to converge, it is necessary that $\lim a_n = 0$).

(b) Converges by alternating series test

(c) Let $a_n = \frac{n^2+4n}{n^3+1}$; then for large n, $a_n \approx \frac{n^2}{n^3} = \frac{1}{n}$. More precisely, if we let $b_n = \frac{1}{n}$, then $\lim a_n/b_n = 1$. Since the series $\sum b_n = \sum \frac{1}{n}$ diverges, by limit comparison test, $\sum a_n$ also diverges.

(d) We use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{e^{2n+2}}{e^{2n}} \cdot \frac{(2n)!}{(2n+2)!} = \frac{n+1}{n} \frac{e^2}{(2n+1)(2n+2)}$$

Thus, $\lim |a_{n+1}/a_n| = 0$, so the series converges.

4. (a) Find the radius of convergence of the following power series.

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^n}{3^n}$$

(b) Write the indefinite integral

$$F(x) = \int f(x)\, dx$$

as a power series (inside the interval of convergence of the power series for $f(x)$.)

(c) The product $\cos(x)f(x)$ can also be written by a power series. Write the first three terms of this power series, up to x^2 term.

Answer:

Using the ratio test:

$$\frac{a_{n+1}}{a_n} = (-1)\frac{(2n+3)x^{n+1}}{3^{n+1}}\frac{3^n}{(2n+1)x^n} = (-1)\frac{(2n+3)}{(2n+1)}\frac{x}{3}$$

so $\lim |a_{n+1}/a_n| = |x/3|$. Therefore, the series converges for $|x| < 3$ and diverges for $|x| > 3$. Therefore, the radius of convergence is $R = 3$.

The indefinite integral can be computed term-by-term:

$$\int \sum (-1)^n \frac{(2n+1)x^n}{3^n} = \sum \int (-1)^n \frac{(2n+1)x^n}{3^n}$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)}{3^n}\frac{x^{n+1}}{n+1}$$

Finally, using only terms up to degree two:

$$f(x) = 1 - \frac{3x}{3} + \frac{5x^2}{9} + \cdots = 1 - x + \tfrac{5}{9}x^2 + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \ldots$$

$$\cos(x)f(x) = \left(1 - \frac{x^2}{2} + \ldots\right)\left(1 - x + \tfrac{5}{9}x^2 + \ldots\right) = 1 - x + \tfrac{5}{9}x^2 - \frac{x^2}{2} + \ldots$$

$$= 1 - x + \frac{x^2}{18} + \ldots$$

where dots stand for terms of degree three and higher.

5. Let $f(x) = \tan^{-1}(2x) - 2x$.

(a) Write the Taylor polynomial $T_5(x)$ of degree 5 for $f(x)$, centered at $x = 0$.

(b) Use the Taylor polynomial found in the previous part to give an approximate value of $f(0.1)$.

Answer:

It is proved in the textbook that

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Thus,

$$\tan^{-1}(2x) - 2x = \left(2x - \frac{(2x)^3}{3} + \frac{(2x)^5}{5} + \dots\right) - 2x = -\frac{8x^3}{3} + \frac{32x^5}{5} + \dots$$

Therefore,

$$T_5(x) = -\frac{8x^3}{3} + \frac{32x^5}{5}.$$

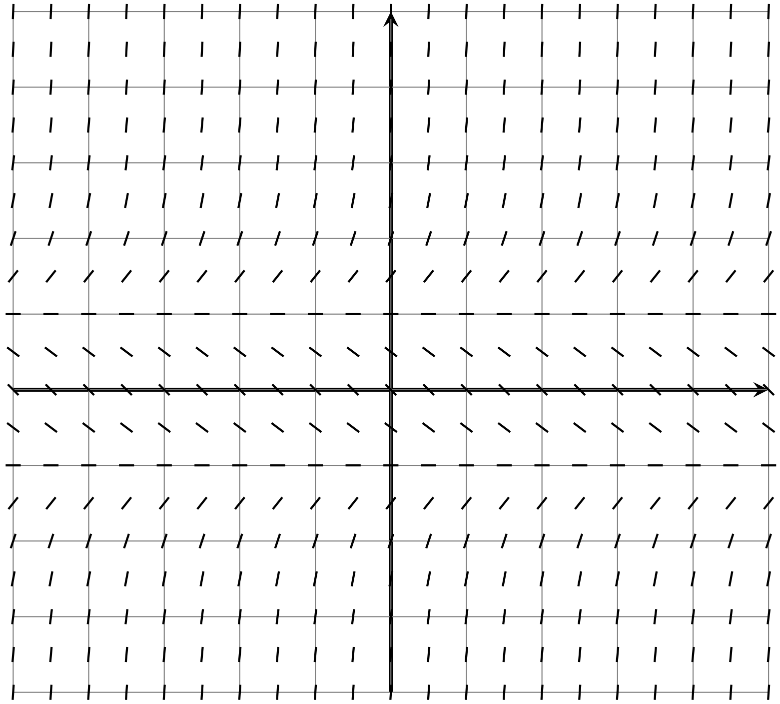
Using it, we write

$$f(0.1) \approx T_5(0.1) = -\frac{8(0.1)^3}{3} + \frac{32(0.1)^5}{5} \approx -0.00260$$

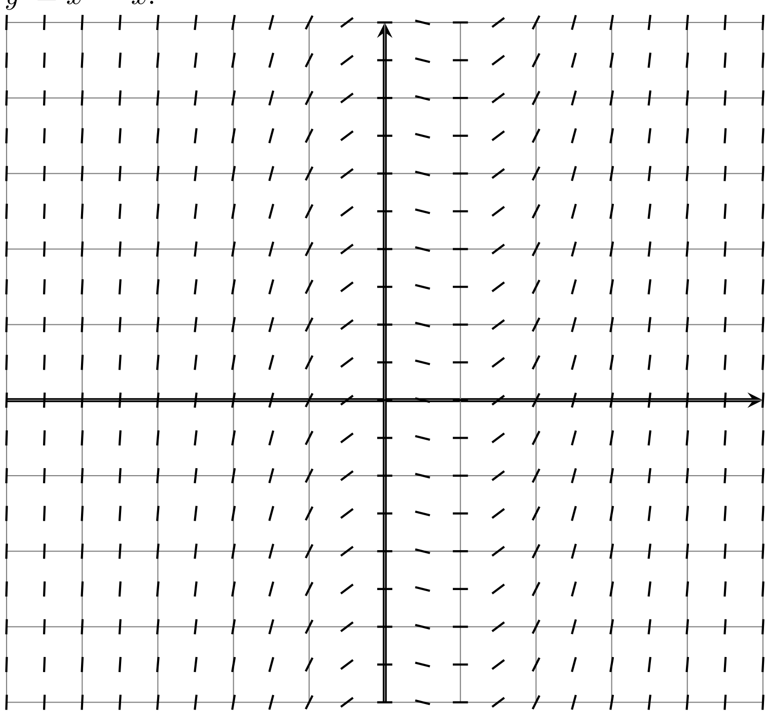
[Note: in the actual exam, since calculators are not allowed, it woudl be OK to leave the answer in the form $-\frac{8(0.1)^3}{3} + \frac{32(0.1)^5}{5}$.]

6. Draw direction fields for the following differential equations.

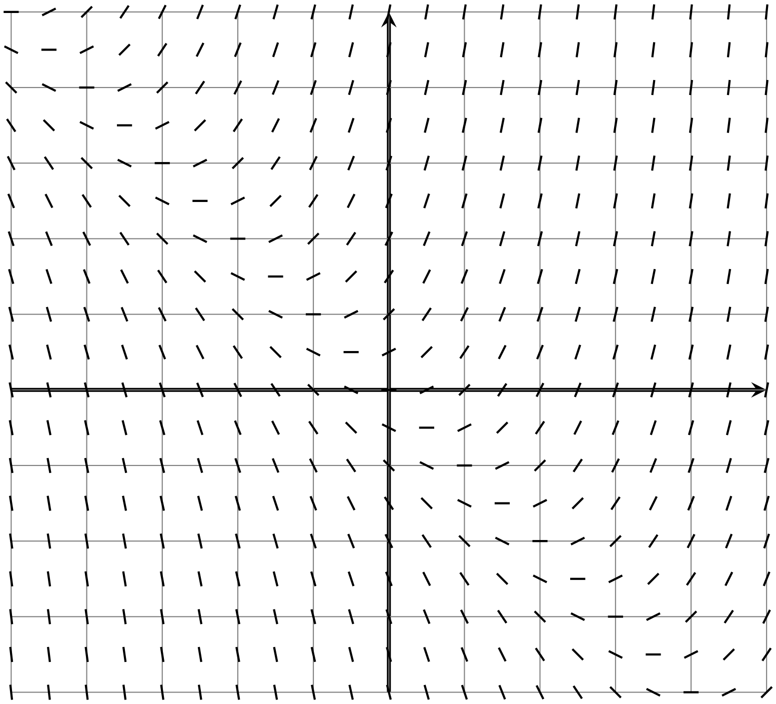
(a) $y' = y^2 - 1$



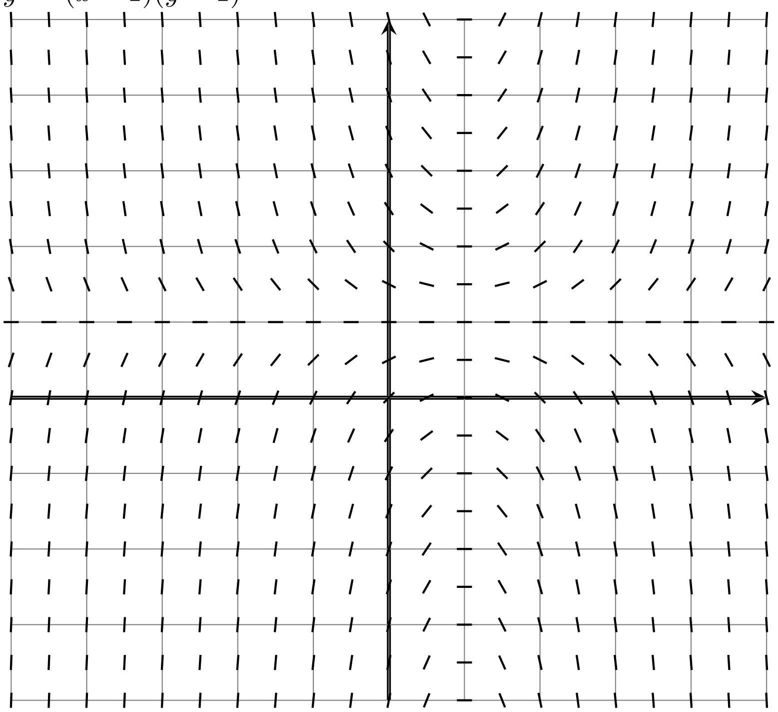
(b) $y' = x^2 - x$.



(c) $y' = x + y$.



(d) $y' = (x-1)(y-1)$



7. Use Eulers Method with step size 0.01 to estimate $y(0.04)$ where y satisfies: $y' = 3y, \quad y(0) = 2$.

Answer:

$$y_0 = 2$$
$$y_1 \approx y_0 + 0.01 \cdot (3y_0) = 2 + 0.01 \cdot 6 = 2.06$$
$$y_2 \approx y_1 + 0.01 \cdot (3y_1) = 2.06 + 0.01 \cdot 3 \cdot 2.06 = 2.1218$$
$$y_3 \approx y_2 + 0.01 \cdot (3y_2) = 2.12.18 + 0.01 \cdot 3 \cdot 2.1218 \approx 2.1855$$
$$y_4 \approx y_3 + 0.01 \cdot (3y_3) = 2.1855 + 0.01 \cdot 3 \cdot 2.1855 \approx 2.2511$$

8. Find the orthogonal trajectories of the family of curves $y^2 = kx^3$.

Answer:

For the original family of curves, the slope at point (x, y) can be found by implicit differentiation:

$$2y\,dy = 3kx^2\,dx$$
$$\frac{dy}{dx} = \frac{3kx^2}{2y} = \frac{3x^2(y^2/x^3)}{2y} = \frac{3y}{2x}$$

where we used $k = y^2/x^3$ to get rid of k in the formula.

Since the slope of the orthogonal curve is negative reciprocal of the slope of the original curve, for the orthogonal family we get

$$\frac{dy}{dx} = -\frac{2x}{3y}$$
$$3y\,dy = -2x\,dx$$
$$\tfrac{3}{2}y^2 = -x^2 + C$$
$$3y^2 + 2x^2 = 2C$$

9. Solve the following initial value problems:

(a) $\frac{dy}{dx} = x^2(y^2+2y-3)$, $y(0) = 2$.

Answer:

$$\frac{dy}{y^2+2y-3} = x^2\,dx$$
$$\int \frac{dy}{(y+3)(y-1)} = \int x^2\,dx + C$$
$$\int \frac{dy}{(y+3)(y-1)} = \tfrac{1}{3}x^3 + C$$

To compute the intergal in the left hand side, use partial fractions:

$$\frac{1}{(y+3)(y-1)} = \frac{A}{y+3} + \frac{B}{y-1}$$

To find A, B, multiply both sides by $(y+3)(y-1)$; this gives

$$A(y-1) + B(y+3) = 1$$
$$(A+B)y + 3B - A = 1$$
$$A + B = 0, \qquad 3B - A = 1$$

Solving this gives $B = \frac{1}{4}$, $A = -\frac{1}{4}$, so

$$\int \frac{dy}{(y+3)(y-1)} = \int \Big(\frac{-1/4}{y+3} + \frac{1/4}{y-1}\Big)\,dy = \frac{1}{4}\Big(-\ln|y+3| + \ln|y-1|\Big)$$
$$= \frac{1}{4}\ln\Big|\frac{y-1}{y+3}\Big|$$

Thus, we get

$$\frac{1}{4}\ln\Big|\frac{y-1}{y+3}\Big| = \tfrac{1}{3}x^3 + C$$
$$\ln\Big|\frac{y-1}{y+3}\Big| = \tfrac{4}{3}x^3 + 4C$$
$$\frac{y-1}{y+3} = \pm e^{\frac{4}{3}x^3+4C} = Ke^{\frac{4}{3}x^3}$$
$$(y-1) = (y+3)Ke^{\frac{4}{3}x^3}$$
$$y(1 - Ke^{\frac{4}{3}x^3}) = 1 + 3Ke^{\frac{4}{3}x^3}$$
$$y = \frac{1 - Ke^{\frac{4}{3}x^3}}{1 + 3Ke^{\frac{4}{3}x^3}}$$

for some constant K. To find K, we use the initial condition $y(0) = 2$, which gives

$$2 = \frac{1-K}{1+3K}$$

Solving it, we get $K = 0.2$, so the final answer is

$$y = \frac{1 - 0.2e^{\frac{4}{3}x^3}}{1 + 0.6e^{\frac{4}{3}x^3}}.$$

(b) $y' = y^2 - 1, \quad y(0) = 2$

Answer:

$$\frac{1}{y^2-1}y' = 1.$$

Hence:

$$\int \frac{1}{y^2-1}dy = \int 1dx,$$
$$\int \frac{1}{(y-1)(y+1)}dy = x + C,$$

Hence:

$$\frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = x + C,$$
$$\frac{1}{2}\ln(\frac{y-1}{y+1}) = x + C,$$
$$\frac{y-1}{y+1} = Ae^{2x}$$

for some constant A. Hence

$$y - 1 = Ae^{2x}(y+1),$$
$$y - 1 = Ae^{2x}y + Ae^{2x},$$
$$y = 1 + Ae^{2x}y + Ae^{2x},$$
$$y - Ae^{2x}y = 1 + Ae^{2x},$$
$$(1 - Ae^{2x})y = 1 + Ae^{2x},$$
$$y = \frac{1 + Ae^{2x}}{1 - Ae^{2x}}.$$

Now $y(0) = 2$ and so

$$y(0) = \frac{1 + Ae^{2\times 0}}{1 - Ae^{2\times 0}},$$
$$y(0) = \frac{1 + A}{1 - A} = 2.$$

Hence:

$$1 + A = 2(1 - A),$$
$$1 + A = 2 - 2A$$
$$1 + A + 2A = 2$$
$$1 + 3A = 2$$
$$3A = 2 - 1$$
$$A = \frac{1}{3}.$$

Hence

$$y = \frac{1 + \frac{1}{3}e^{2x}}{1 - \frac{1}{3}e^{2x}}.$$

(c) $y' = \frac{1}{y(\sqrt{1-x^2})}, y(0) = 1.$

Answer:

$$\begin{aligned} yy' &= \frac{1}{\sqrt{1-x^2}}, \\ \int y\,dy &= \int \frac{1}{\sqrt{1-x^2}}dx, \\ \frac{y^2}{2} &= \arcsin(x) + C, \\ y^2 &= 2\arcsin(x) + 2C, \\ y &= \sqrt{2\arcsin(x) + 2C}. \end{aligned}$$

Now $y(0) = 1$ and so:

$$\begin{aligned} 1 &= \sqrt{2\arcsin(0) + 2C}, \\ 1 &= \sqrt{2C} \\ 1 &= 2C. \end{aligned}$$

Hence

$$y = \sqrt{2\arcsin(x) + 1}.$$

10. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

Answer:

Let $S(t)$ be the amount of salt (in kg) at time t (measured in minutes). Then the concentration of salt (in kg/L) is $S(t)/100$. This gives the differential equation

$$\frac{dS}{dt} = (0.1) \cdot 10 - \frac{S(t)}{100} \cdot 10 = 1 - 0.1S(t)$$

This is a separable equation:

$$\begin{aligned}
&\frac{dS}{1 - 0.1S} = dt \\
&\frac{10\,dS}{10 - S} = dt \\
&\int \frac{10\,dS}{10 - S} = t + C \\
&-10\ln(10 - S) = t + C \\
&\ln(10 - S) = -\frac{t + C}{10} \\
&10 - S = e^{-(t+C)/10} = Ae^{-t/10} \\
&S = 10 - Ae^{-t/10}
\end{aligned}$$

At $t = 0$, we have $S = 0$, so $10 - Ae^0 = 0$. Thus, $A = 10$, so

$$\begin{aligned}
&S(t) = 10 - 10e^{-t/10} \\
&S(6) = 10 - 10e^{-0.6} \approx 4.512
\end{aligned}$$

11. A sample of tritium-3 decayed to 94.5% of its original amount after a year.

(a) What is the half-life of Tritium-3?

(b) How long would it take the sample to decay to 20% of its original amount?

Answer:

The equation of radioactive decay is

$$M(t) = M(0)e^{-kt}$$

where $M(t)$ is amount after time t, and k depends on the material.

From the information given, $M(1) = 0.945M(0)$, so

$$e^{-k} = 0.945$$

Solving it, we get

$$k = -\ln(0.945) \approx 0.05657$$

Half-life T is determined by $M(T) = \frac{1}{2}M(0)$, i.e.

$$e^{-kT} = \frac{1}{2}$$

which gives $-kT = \ln(1/2)$, or

$$T = \ln(2)/k \approx 12.25 \text{ years}$$

To find the time it takes to decay to 20%, we need to solve

$$e^{-kt} = 0.2$$

or

$$t = -\ln(0.2)/k \approx 28.45 \text{ years}$$

12. A population of bees in a particular region satisfies the logistic equation with carrying capacity 10000. Suppose that there are only 1000 bees initially and 2000 bees after 2 years. How many bees are there after 3 years?

Answer:
Let P be the population of bees and t the time in years. Then

$$P' = kP(1 - \frac{P}{10000}).$$

The solution to this equation is:

$$P = \frac{10000}{1 + Ae^{-kt}}.$$

There are initially 1000 bees and so:

$$P(0) = \frac{10000}{1 + A} = 1000.$$

Therefore:

$$\begin{aligned} \frac{10000}{1+A} &= 1000, \\ 1 + A &= \frac{10000}{1000}, \\ 1 + A &= 10, \\ A &= 9. \end{aligned}$$

Therefore:

$$P = \frac{10000}{1 + 9e^{-kt}}.$$

Also, $P(2) = 2000$ and so:

$$\begin{aligned} P(2) = \frac{10000}{1 + 9e^{-2k}} &= 2000, \\ \frac{10000}{1 + 9e^{-2k}} &= 2000, \\ 1 + 9e^{-2k} &= \frac{10000}{2000}, \\ 1 + 9e^{-2k} &= 5, \\ 9e^{-2k} &= 4, \\ e^{-2k} &= \frac{4}{9}, \\ -2k &= \ln \frac{4}{9}, \\ -2k &= -\ln \frac{9}{4}, \\ 2k &= \ln \frac{9}{4}, \\ k &= \frac{1}{2} \ln \frac{9}{4}. \end{aligned}$$

Therefore:

$$P = \frac{10000}{1 + 9e^{-\frac{1}{2}\ln\frac{9}{4}t}}.$$

Hence

$$P(3) = \frac{10000}{1 + 9e^{-\frac{1}{2}\ln\frac{9}{4}\times 3}}.$$

Hence there are

$$P(3) = \frac{10000}{1 + 9e^{-\frac{3}{2}\ln\frac{9}{4}}} \approx 2727$$

bees after 3 years.

13. Find all equilibrium solutions of the following system of differential equations:

$$\frac{dW}{dt} = R^2 + RW$$
$$\frac{dR}{dt} = W^2 - R.$$

Answer:
We need to solve:

$$0 = R^2 + RW$$
$$0 = W^2 - R.$$

Hence:

$$0 = R(R + W)$$

and so $R = 0$ or $R = -W$. If $R = 0$ then $0 = W^2$ and so $W = 0$. If $R = -W$ then $W^2 - (-W) = 0$ and so $W^2 + W = 0$ and so $W(W + 1) = 0$ and hence $W = 0$ or -1. If $W = 0$ then $R = -W = 0$ and if $W = -1$ then $R = 1$.

Therefore:

$$(R, W) = (0, 0), \quad (R, W) = (1, -1)$$

are the equilibrium solutions.