Practice Midterm 2 (corrected)

MAT 126

October 2012

Name: (please print)	**ID #**:
Your recitation: (see list below)	

Lec. 1	**TuTh 10am**	**Michael Movshev**
R01	F 10am	Matthew Wroten
R02	M 10am	Jan Gutt
R03	Tu 1pm	Jan Gutt
R04	Th 4pm	Chengjian Yao
R05	W 5:30pm	Chengjian Yao
Lec. 2	**MWF 10am**	**Alexander Kirillov**
R06	M 12pm	Claudio Meneses
R07	Th 10am	Mark Flanagan
R08	Tu 8:30am	Matthew Wroten
R10	W 11am	Claudio Meneses
Lec. 3	**TuTh 5:30pm**	**Ming-Tao Chuan**
R13	M 4pm	Kirill Lazebnik
R14	Th 2:30pm	Mark Flanagan
R16	Th 7pm	Kirill Lazebnik

No notes, books or calculators.

You must show your reasoning, not just the answer. Answers without justification will get only partial credit.

Please cross out anything that is not part of your solution — e.g., some preliminary computations that you didn't need.

All answers should be simplified if possible — e.g., sin(0) should be replaced by 0. However, unless instructed, do not replace exact answers by approximate ones — e.g. do not replace $\sqrt{2}$ by 1.41

Each problem is worth 10 pts.

1. Find the derivative of the following function:

$$s(x) = \int_{\frac{1}{2}\sin(x)}^{\frac{1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

Solution

Let

$$f(x) = \int_{x}^{\frac{1}{2}} \frac{dt}{\sqrt{1-t^2}} = -\int_{\frac{1}{2}}^{x} \frac{dt}{\sqrt{1-t^2}}$$

By FTC, $f'(x) = -\frac{1}{\sqrt{1-x^2}}$.

Since $s(x) = f\left(\frac{1}{2}\sin(x)\right)$, chain rule gives

$$s'(x) = f'\left(\frac{1}{2}\sin(x)\right) \cdot \frac{1}{2}\cos(x) = -\frac{1}{\sqrt{1-\frac{1}{4}\sin^2(x)}} \cdot \frac{1}{2}\cos(x)$$

2. Evaluate the following indefinite integrals:

(a)

$$\int x^5 \ln(x) dx$$

(b)

$$\int \frac{\cos^3(x)}{\sin(x)}\, dx$$

Solution

(a) Integration by parts, with $f = \ln(x)$, $g(x) = \frac{1}{6}x^6$ (so that $g'(x) = x^5$):

$$\begin{aligned}\int x^5 \ln(x) dx &= \tfrac{1}{6}x^6 \ln(x) - \int \tfrac{1}{6}x^6 \cdot \frac{1}{x} dx \\ &= \tfrac{1}{6}x^6 \ln(x) - \int \tfrac{1}{6}x^5 dx \\ &= \tfrac{1}{6}x^6 \ln(x) - \tfrac{1}{36}x^6 + C\end{aligned}$$

(b) Use the substitution $u = \sin(x)$, $du = \cos(x)\, dx$. Then:

$$\begin{aligned}\int \frac{\cos^3(x)}{\sin(x)}\, dx = \int \frac{\cos^2(x)}{\sin(x)} \cos(x)\, dx &= \int \frac{1 - \sin^2(x)}{\sin(x)} \cos(x)\, dx \\ &= \int \frac{1 - u^2}{u}\, du = \int (u^{-1} - u)\, du \\ &= \ln(|u|) - \frac{1}{2}u^2 + C \\ &= \ln(|\sin(x)|) - \frac{1}{2}\sin^2(x) + C\end{aligned}$$

3. Evaluate the following definite integrals:

(a)

$$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^3} dx$$

(b)

$$\int_{0}^{2} x^2\sqrt{4-x^2}dx$$

(c)

$$\int_{1}^{e^\pi} \frac{\cos(\ln x)\sin^2(\ln x)}{x} dx$$

(d)

$$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2} dx$$

Solution

(a) Substitution $u = 1/x$, $du = -\frac{1}{x^2}\,dx$ gives

$$\begin{aligned}\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^3} dx &= \int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x}\frac{dx}{x^2} \\ &= \int_{\pi}^{\pi/2} -u\sin u\,du\end{aligned}$$

Now use integration by parts: since $-\sin(u) = \cos'(u)$,

$$\begin{aligned}\int_{\pi}^{\pi/2} -u\sin u\,du &= u\cos(u)\Big|_{\pi}^{\pi/2} - \int_{\pi}^{\pi/2} \cos(u)\,du \\ &= (u\cos(u) - \sin(u))\Big|_{\pi}^{\pi/2} \\ &= \tfrac{\pi}{2}\cos(\pi/2) - \sin(\pi/2) - (\pi\cos(\pi) - \sin(\pi)) = -1+\pi = \pi - 1\end{aligned}$$

(b) Rewrite

$$\sqrt{4-x^2} = \sqrt{4(1-x^2/4)} = 2\sqrt{1-\left(\tfrac{x}{2}\right)^2}.$$

Now, use trigonometric substitution $\frac{x}{2} = \sin(t)$, so

$$\begin{aligned}&\sqrt{1-\left(\tfrac{x}{2}\right)^2} = \sqrt{1-\sin^2(t)} = \cos(t) \\ &dx = 2\cos(t)dt\end{aligned}$$

This gives

$$\begin{aligned}\int_{0}^{2} x^2\sqrt{4-x^2}dx &= \int_0^{\pi/2} (2\sin(t))^2(2\cos(t))(2\cos(t)dt) \\ &= \int_0^{\pi/2} 16\sin^2(t)\cos^2(t)\,dt = \int_0^{\pi/2} 16(\sin(t)\cos(t))^2\,dt\end{aligned}$$

Since $\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$, we get

$$\int_0^{\pi/2} 4\sin^2(2t)\,dt$$

Using trigonometric identity $\sin^2(a) = \frac{1-\cos(2a)}{2}$, this can be rewritten as

$$\int_0^{\pi/2} 2(1-\cos(4t))\,dt = 2t - \frac{\sin(4t)}{2}\Big|_0^{\pi/2} = \pi.$$

(c) Let $u = \sin(\ln(x))$. Then $du = \cos(\ln(x)) \cdot \frac{1}{x}$, and $x = 1$ gives $u = \sin(0) = 0$, and $x = e^\pi$ gives $u = \sin(\pi) = 0$, so

$$\int_1^{e^\pi} \frac{\cos(\ln x)\sin^2(\ln x)}{x}dx = \int_0^0 u^2\,du = 0$$

(d) Same substitution as in part (a):

$$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{x^2}dx = \int_{1/\pi}^{2/\pi} \sin(1/x)\frac{dx}{x^2} = \int_\pi^{\pi/2} -\sin u\,du = \cos(u)\Big|_\pi^{\pi/2} = 1$$

4. Evaluate the integrals

(a)

$$\int_0^1 \frac{9}{x^2+3}dx$$

(b)

$$\int_0^1 \frac{x+1}{x^2-9}dx$$

Solution

(a) Use subsitution $x = \sqrt{3}u$, $u = \frac{1}{\sqrt{3}}x$:

$$\int_0^1 \frac{9}{x^2+3}dx = \int_0^{1/\sqrt{3}} \frac{9\sqrt{3}du}{3u^2+3} = \frac{9\sqrt{3}}{3}\int_0^{1/\sqrt{3}} \frac{du}{u^2+1}$$
$$= 3\sqrt{3}\tan^{-1}(u)\Big|_0^{1/\sqrt{3}} = 3\sqrt{3}\tan^{-1}(1/\sqrt{3})$$
$$= 3\sqrt{3}\cdot\pi/6 = \frac{\sqrt{3}\pi}{2}$$

(b) Use partial fractions: $x^2-9=(x-3)(x+3)$, so we can write

$$\frac{x+1}{(x-3)(x+3)} = \frac{A}{x-3}+\frac{B}{x+3}$$

We now solve for A, B:

$$x+1 = A(x+3)+B(x-3) = (A+B)x+3A-3B$$
$$A+B=1$$
$$3A-3B=1$$

Solving the last system of 2 equations in two variables, we get $A=2/3$, $B=1/3$. Thus,

$$\int_0^1 \frac{x+1}{x^2-9}dx = \int_0^1 \left(\frac{2/3}{x-3}+\frac{1/3}{x+3}\right)dx = \left(\tfrac{2}{3}\ln(|x-3|)+\tfrac{1}{3}\ln(|x+3|)\right)\Big|_0^1$$
$$= (\tfrac{2}{3}\ln|1-3|+\tfrac{1}{3}\ln(1+3)) - (\tfrac{2}{3}\ln|0-3|+\tfrac{1}{3}\ln|0+3|)$$
$$= \tfrac{2}{3}\ln(2)+\tfrac{1}{3}\ln(4)-\tfrac{2}{3}\ln(3)-\tfrac{1}{3}\ln(3)$$
$$= \tfrac{4}{3}\ln(2)-\ln(3)$$

(since $\ln(4) = 2\ln(2)$).

5. (a) Decompose the rational function into partial fractions

$$\frac{x^3 - 2x^2 - 7x + 10}{x^2 - 5x + 6}$$

(b) Compute the integral

$$\int \frac{x^3 - 2x^2 - 7x + 10}{x^2 - 5x + 6}\, dx$$

Solution

(a) Doing the long division gives

$$\frac{x^3 - 2x^2 - 7x + 10}{x^2 - 5x + 6} = x + 3 + \frac{2x - 8}{x^2 - 5x + 6}$$

Factoring gives $x^2 - 5x + 6 = (x - 2)(x - 3)$, so we now use partial fractions:

$$\frac{2x - 8}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}$$

$$2x - 8 = A(x - 3) + B(x - 2) = (A + B)x - 3A - 2B$$

$$A + B = 2$$

$$-3A - 2B = -8$$

Solving the last system gives $A = 4, B = -2$, so the final answer is

$$\frac{x^3 - 2x^2 - 7x + 10}{x^2 - 5x + 6} = x + 3 + \frac{4}{x - 2} - \frac{2}{x - 3}$$

(b) Using part (a), we get:

$$\int \frac{x^3 - 2x^2 - 7x + 10}{x^2 - 5x + 6}\, dx = \int x + 3 + \frac{4}{x - 2} - \frac{2}{x - 3}\, dx$$

$$= \tfrac{1}{2}x^2 + 3x + 4\ln(|x - 2|) - 2\ln(|x - 3|) + C$$