EARLY EXAM
MAT 125 and 131
September 19, 2002, 8:30-10 p.m.

Correct answers are boxed.

1. Which of the following straight lines is parallel to the line $2y = 4x - 3$?

   (a) $2x = 4y + 7$
   (b) $4x = -2y + 3$
   (c) $2y + 4x = -3$
   $\boxed{\text{(d)}}$ $y = 2x + 2$
   (e) none of these

   Solution: Rewriting the equation of the original line as $y = 2x - 1.5$, we see that the slope is 2. Similarly, rewriting equations (a)–(d) in the form $y = mx + a$, we see that the slope m for these lines is

   (a): $m = 1/2$
   (b): $m = -2$
   (c): $m = -2$
   (d): $m = 2$

   Since two lines are parallel if and only if they have the same slope, the answer is (d).

2. Which of the following is the equation of the straight line passing through the points $(-1, 0)$ and $(1, 1)$?

   $\boxed{\text{(a)}}$ $y = \dfrac{x+1}{2}$
   (b) $y = \dfrac{x-1}{2}$
   (c) $y = \dfrac{x}{2} + 1$
   (d) $y + x = 1$
   (e) none of these

   Solution: Equation of the line passing through points (x_1, y_1) and x_2, y_2 is

$$y - y_1 = m(x - x_1), \qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

which for $x_1 = -1, y_1 = 0, x_2 = 1, y_2 = 1$ gives

$$m = \frac{1-0}{1-(-1)} = 1/2 \quad y = \frac{1}{2}(x+1).$$

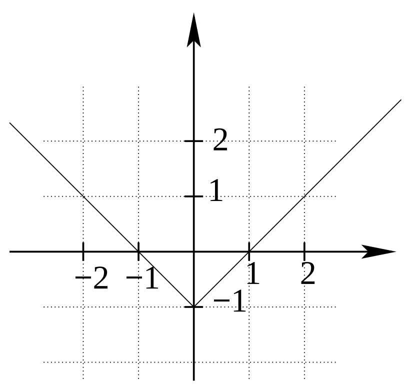
3. $\dfrac{3^{3x}}{9^{(-x/4)}} =$

   (a) $3^{10x/3}$
   $\boxed{\text{(b)}}$ $3^{7x/2}$
   (c) $3^{5x/2}$
   (d) $\left(\frac{1}{3}\right)^{3x}$
   (e) none of these

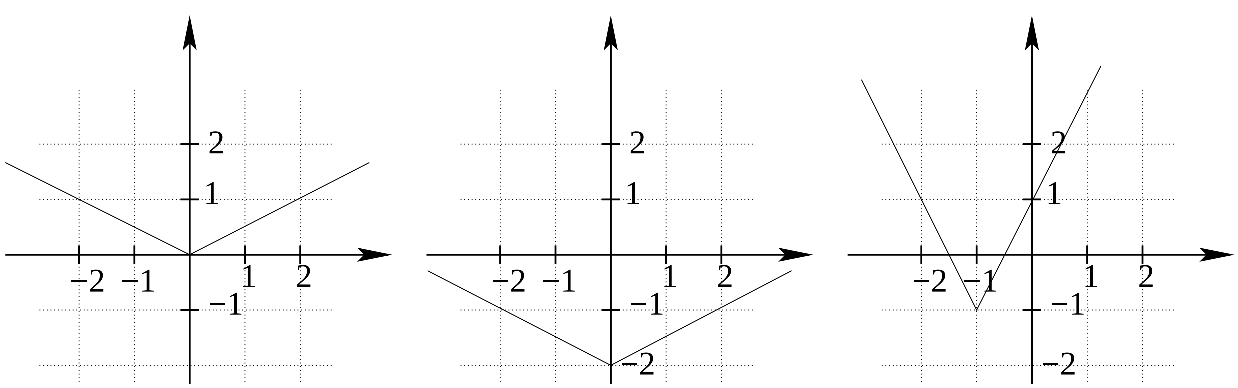
   Solution:

$$\frac{3^{3x}}{9^{(-x/4)}} = 3^{3x}9^{x/4} = 3^{3x}3^{2x/4} = 3^{3x+x/2} = 3^{7x/2}$$

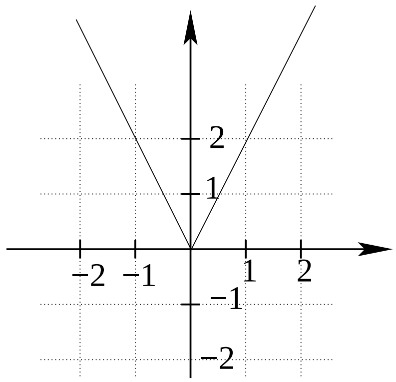
4. The following is a graph of a function $f(x)$.



   Which of the following graphs is the graph of $f(2x)+1$?



(a) (b) (c)



$\boxed{\text{(d)}}$ (e) none of these

Solution: The graph of $f(2x)+1$ is obtained from the graph of $f(x)$ by (1)compressing by factor of 2 in the horizontal direction and then (2) moving 1 unit up.

5. If $x = \log_2 3$, then $8^{x/2} =$
(a) 3^3
$\boxed{\text{(b)}}$ $3\sqrt{3}$
(c) 2^3
(d) $2^{3.5}$
(e) none of these

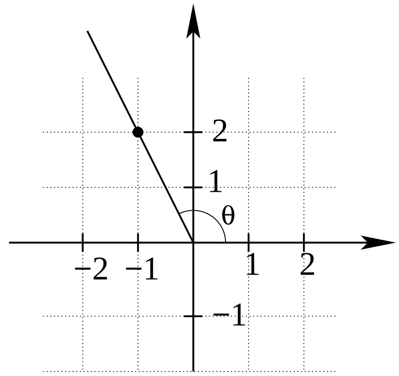
Solution:
$$8^{x/2} = (2^3)^{x/2} = 2^{3x/2} = (2^x)^{3/2}$$
By definition of logarithm, $2^x = 2^{\log_2 3} = 3$, so we get $3^{3/2} = 3^1 \cdot 3^{1/2} = 3\sqrt{3}$.

6. Let $p(x) = (x+7)^2 + 3$. Then $p(x)$ is smallest when $x =$
(a) 0
(b) 7
$\boxed{\text{(c)}}$ -7
(d) -3
(e) none of these

Solution: Since $(x+7)^2 \geq 0$, we see that $p(x) \geq 3$, and equality can only hold if $(x+7)^2 = 0$, i.e. $x = -7$

7. Let θ denote the angle in the following picture. What is $\sin(\theta)$?

(a) $\dfrac{-1}{2}$
(b) -2
$\boxed{\text{(c)}}$ $\dfrac{2}{\sqrt{5}}$
(d) $\dfrac{2}{5}$
(e) none of these



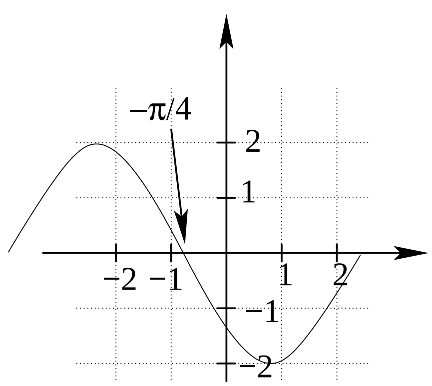
Solution: This line contains the point $(-1, 2)$. The distance from this point to the origin is $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$. Using the formula $y = r \sin\theta$, we see that $\sin\theta = y/r = 2/\sqrt{5}$.

8. Which of the following is the set of all solutions of the inequality $x^2 + 2x - 3 \leq 0$?
$\boxed{\text{(a)}}$ $[-3, 1]$
(b) $[-3, 3]$
(c) $(-\infty, -3] \cup [1, \infty)$
(d) $[1, \infty)$
(e) none of these

Solution: First, graph of the function $x^2 + 2x - 3$ is a parabola with branches going up. Solutions of $x^2 + 2x - 3 \leq 0$ are those values of x for which the graph is below x axis. From the graph it is obvious that this is exactly the interval between the roots (if there are roots).

To find the roots, we factor $x^2 + 2x - 3 = (x + 3)(x - 1)$, so the roots are $x = -3, x = 1$ and the interval between the roots is $[-3, 1]$ (including endpoints because we have non-strict inequality.

9. The following is the graph of the function $A\sin(x + b)$ (x is measured in radians). What are A, b?



(a) $A = 1/2, b = -\pi/4$
$\boxed{\text{(b)}}$ $A = -2, b = \pi/4$
(c) $A = 2, b = -\pi/4$
(d) $A = 2, b = \pi/4$
(e) none of these

Solution: This graph is obtained from the graph of $\sin x$ by moving $\pi/4$ units to the left, reflecting around the x axis, and then stretching vertically by factor of 2. These transformations correspond to replacing $\sin x$ by $\sin(x + \pi/4)$, then multiplying by -1, which gives $-\sin(x + \pi/4)$, and then multiplying by 2, which gives

$$2 \cdot (-\sin(x + \pi/4)) = -2\sin(x + \pi/4).$$

10. Which of the following is the set of **all** solutions of the equations $\log_3 x + \log_3(x-1) = 0$

    (a) Two solutions: $x = 0, x = 1$
    (b) One solution: $x = 1$
    $\boxed{\text{(c)}}$ One solution: $x = (1 + \sqrt{5})/2$
    (d) Two solutions: $x = (1 + \sqrt{5})/2, x = (1 - \sqrt{5})/2$
    (e) No solutions

    Solution: Using $\log_3(x) + \log_3(x-1) = \log_3(x(x-1))$ and $\log_3 1 = 0$, we rewrite this equation as

    $$\begin{aligned} \log_3(x(x-1)) &= \log_3(1) \\ x(x-1) &= 1 \\ x^2 - x - 1 &= 0 \end{aligned}$$

    Using the formula for the solutions of a quadratic equation, we get the roots of this equation: $x = (1 + \sqrt{5})/2, x = (1 - \sqrt{5})/2$.

    However, there is a catch. The formula $\log_3(x) + \log_3(x-1) = \log_3(x(x-1))$ we used is only valid if both $x, x-1$ are positive. But $x = (1 - \sqrt{5})/2 < 0$, so for this value of x, $\log_3 x$ is not defined, so it is not a solution of the original equation. For $x = (1 + \sqrt{5})/2$, both $x, x-1$ are positive, so all our arguments are valid. Thus, $x = (1 + \sqrt{5})/2$ is the only solution.

11. Which of the following is the set of **all** solutions of inequality $2^{-x} < 4$?
    (a) $x < -2$
    $\boxed{\text{(b)}}$ $x > -2$
    (c) $0 < x < -2$
    (d) $x > 2$
    (e) none of the above

    Solution: We need to find for which values of x the graph of 2^{-x} is below the line $y = 4$. This line crosses the graph at the point where $2^{-x} = 4$), i.e. $x = -2$. Since the function 2^{-x} is decreasing, it is clear from the graph that solutions of the inequality $2^{-x} < 4$ are all points to the right of the point $x = -2$, i.e. the interval $(-2, +\infty)$.

12. Let $f(x) = \sqrt{x}$, and let $g(x) = x^2 + 1$. What is $f(g(-1))$?

    (a) 0
    $\boxed{\text{(b)}}$ $\sqrt{2}$

(c) 1
(d) -1
(e) undefined

Solution: $g(-1) = (-1)^2 + 1 = 2$, so $f(g(-1)) = f(2) = \sqrt{2}$.

13. The function $h(x) = \dfrac{(x+1)^2 + 1}{x}$ can be written as the following composition:
(a) $h(x) = f(g(x)), \quad f(x) = x + 1, \quad g(x) = \frac{(x+1)^2}{x}$
(b) $h(x) = f(g(x)), \quad f(x) = \frac{(x+1)^2}{x}, \quad g(x) = x + 1$
(c) $h(x) = f(g(x)), \quad f(x) = x + 1, \quad g(x) = \frac{x^2+1}{x-1}$
$\boxed{\text{(d)}}$ $h(x) = f(g(x)), \quad f(x) = \frac{x^2+1}{x-1}, \quad g(x) = x + 1$
(e) none of these

Solution: Let us calculate each of these compositions:
(a) $f(g(x)) = f(\frac{(x+1)^2}{x}) = \frac{(x+1)^2}{x} + 1 = \frac{(x+1)^2+x}{x}$
(b) $f(g(x)) = f(x+1) = \frac{(x+1+1)^2}{x+1} = \frac{(x+2)^2}{x+1}$
(c) $f(g(x)) = f(\frac{x^2+1}{x-1}) = \frac{x^2+1}{x-1} + 1 = \frac{x^2+1+x-1}{x-1}$
(d) $f(g(x)) = f(x+1) = \frac{(x+1)^2+1}{(x+1)-1} = \frac{(x+1)^2+1}{x}$

The next 3 questions are True/False questions. Pick (a) if the statement is true. Pick (b) if the statement is false.

14. For all x, one has $\frac{\cos(2x)}{\cos(x)} = 2$
(a) true
$\boxed{\text{(b)}}$ false

Solution: Take, for example, $x = 0$, then $\cos x = \cos(2x) = 1$

15. If $p(x) = ax^2 + bx + c$, where $a \neq 0$, then p cannot be an increasing function on the whole real line.
$\boxed{\text{(a)}}$ true
(b) false

Solution: The graph of this function is always a parabola, so it contains both going down part (on one side of the vertex) and going up part.

16. If $x > 0$, then $\ln x > 0$.
(a) true
$\boxed{\text{(b)}}$ false

Solution: For example, for $x = 1/e, \ln x = -1 < 0$. More genrally, for all $x \in (0, 1)$, $\ln x < 0$.