Flat Projective Connections

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1. Geometric Quantization

In this section we recall basic points of Kostant's geometric quantization [K]$_2$. We consider a purely holomorphic version, so all the objects below will be algebraic or analytic ones. The language of complex polarizations is discussed in no. 1.7.

1.1 Recollections from Classical Mechanics

1.1.1 Definition. *Let $p : X \to S$ be a smooth morphism of smooth varieties. A connection for p, or simply, or simply, p-connection, is an $\mathcal{O}_X$-linear morphism $\nabla_S : p^*\mathcal{T}_S \to \mathcal{T}_X$ such that $dp \circ \nabla_S = id_{p^*T_S}$; such ∇_S is integrable if the corresponding map $\mathcal{T}_S \to p_*\mathcal{T}_X$ commutes with brackets.* □

Let ∇_S be a p-connection. Then $\nabla_S(p^*\mathcal{T}_S) \subset \mathcal{T}_X$ is a subbundle transverse to fibers of p; we will call it ∇_S-horizontal subbundle. Conversely, any subbundle transverse to fibers of a smooth p defines a p-connection which is integrable iff the subbundle is integrable. For any $s \in S$ an integrable

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p-connection ∇_S defines a trivialization of p over a formal neighborhood $S_s^\wedge$ of s (i.e., the isomorphism $X_{S_s^\wedge} = X_S \times S_s^\wedge$, where $X_S = p^{-1}(s)$, etc.).

Localizing on S we see that p-connections form a sheaf p-conn on S. If ∇ is a p-connection and $\nu \in \mathrm{Hom}(\mathcal{T}_S, p_*\mathcal{T}_{X/S}) = \Omega_S^1 \otimes p_*\mathcal{T}_{X/S}$, then $\nabla + \nu$ is also a p-connection. This way p-conn is an $\Omega_S^1 \otimes p_*\mathcal{T}_{X/S}$-torsor.

Note that an integrable p-connection ∇_S defines an action of $\mathcal{T}_S$ on relative differential forms $\Omega_{X/S}$ (by Lie derivatives along horizontal vector field $\nabla_S(\mathcal{T}_S)$); we will say that a form $\omega \in \Omega^i_{X/S}$ is ∇_S-horizontal if ω is fixed by the $\mathcal{T}_S$-action.

Let (X, ω) be a symplectic variety, i.e., X is a smooth variety and ω is a non-degenerate closed 2-form on X. Then ω defines Poisson brackets $\{\ ,\ \}$ on $\mathcal{O}_X$ in a usual manner.

1.1.2 Definition. *A surjective morphism of varieties $\pi : X \to Y$ is called polarization, or Lagrangian projection, if dim $Y = \frac{1}{2}$ dim X and $\{\ ,\ \}$ vanishes on $\pi^{-1}\mathcal{O}_Y \subset \mathcal{O}_X$; such π is called smooth if π is a smooth morphism.*

□

A basic example of such $\pi : X \to Y$ is a twisted cotangent bundle over Y (see A1.8, A1.9).

1.1.3 Definition. *Let S be a smooth variety. An S-Lagrangian triple consists of a morphism $\pi : X \to Y$ of S-varieties (i.e., one has a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ p_X \searrow & & \swarrow p_Y \\ & S & \end{array}$$

), a relative 2-form $\omega \in \Omega^2_{X/S}$ and a p-connection ∇_S such that

(i) p_X, p_Y *and* π *are smooth surjective morphisms.*

(ii) *a form* ω *is closed and non-degenerate, i.e., for any* $s \in S$ *the fiber* (X_s, ω_s) *is a symplectic variety.*

(iii) *for any* $s \in S$ *the morphism* $\pi_s : X_s \to Y_s$ *is a twisted cotangent bundle over* Y_s*.*

(iv) ∇_S *is integrable and* ω *is* ∇_S*-horizontal.* $\square$

Assume we have a Lagrangian triple (1.1.3). Consider the $\mathcal{O}_Y$-algebra $A := \pi_* \mathcal{O}_X$. It carries $\mathcal{O}_S$-linear Poisson bracket $\{\ ,\ \}$ and a natural filtration A_i such that $A_0 = \mathcal{O}_Y$, $A_i = S^i A_1$ and $\mathrm{gr}.A = S^{\cdot}\mathcal{T}_{Y/S}$ (see A1.8). Our connection ∇_S is an $\mathcal{O}_S$-linear morphism $\nabla_S : \mathcal{T}_S \to \mathrm{Der}\ A$ such that for $f \in \mathcal{O}_S \subset A$, $\tau \in \mathcal{T}_S$ one has $\nabla_S(\tau)(f) = \tau(f)$; according to (iv) ∇_S commutes with brackets and for $a, b \in A$, $\tau \in \mathcal{T}_S$ one has $\nabla_S(\tau)(\{a, b\}) = \{\nabla_S(\tau)(a), b\} + \{a, \nabla_S(\tau)(b)\}$.

Let n be a minimal integer such that $\nabla_S(\tau_S)(A_0) \subset A_n$; such n is called an order of our Lagrangian triple. For example, $n = 1$ means that for any f, $g \in \mathcal{O}_Y$, $\tau \in \mathcal{T}_S$ one has $\{\nabla_S(\tau)(f), g\} \in \mathcal{O}_Y$.

1.1.4 Lemma. (i) *One has* $\nabla_S(\mathcal{T}_S)(A_i) \subset A_{i+n}$ *for any* i*. Hence we have an* $\mathcal{O}_S$*-linear map* $\mathrm{gr}\nabla_S : \mathcal{T}_S \to \mathrm{Der}^{(n)}\mathrm{gr}.A = \mathrm{Der}^{(n)} S^{\cdot}\mathcal{T}_{Y/S}$ *(here* $\mathrm{Der}^{(n)}$ *means differentiations of homogeneous degree* n*).*

(ii) *Assume that* $n \geq 1$*. There exists a unique* $\mathcal{O}_S$*-linear map* $\sigma\nabla_S : \mathcal{T}_S \to S^{n+1}\mathcal{T}_{Y/S}$ *such that* $(\mathrm{gr}\nabla_S(\tau))(f) = \{\sigma\nabla_S(\tau), f\}$ *for* $\tau \in \mathcal{T}_S$, $f \in S^{\cdot}\mathcal{T}_{Y/S}$*. The functions* $\sigma\nabla_S(\tau)$, $\tau \in \mathcal{T}_S$*, Poisson commute.*

Proof: Clear. □

Sometimes it is convenient to describe S-Lagrangian triples in a different language. Let Y be an S-variety such that $p_Y : Y \to S$ is smooth and surjective.

1.1.5 Definition. *An S-Hamiltonian datum on Y consists of*

— *a twisted cotangent bundle* $(\widetilde{X}, \omega_{\widetilde{X}})$, $\tilde{\pi} : \widetilde{X} \to Y$ *over* Y. Put $X := \widetilde{X}$ mod $p_Y^* \Omega_S^1$: *this is a* $T^*_{Y/S}$*-torsor over* Y*; let* $\widetilde{X} \xrightarrow{r} X \xrightarrow{\pi} Y$ *be the projections.*

— *a section* $h : X \to \widetilde{X}$ *of* r *(called Hamiltonian of our datum).*

Put $\omega_X := h^* \omega_{\widetilde{X}}$: this is a closed 2-form on X. The following integrability axiom should hold:

for each $x \in X$ the form $\omega_x \in \Lambda^2 T^*_{X^x}$ has rank dim X-dim S. □

Assume we have a Hamiltonian datum 1.1.5. Note that for each $s \in S$ the map $\pi_s : X_s \to Y_s$ is the induced (from $\widetilde{X}$) twisted cotangent bundle on Y_s (see A3). The symplectic form ω_s coincides with $\omega_{X|X_s}$, so the integrability axiom asserts that ω_X has minimal possible rank (in particular, in case dim $S = 1$ this axiom holds automatically). The kernels of ω_{X^x}, $x \in X$, form a subbundle transversal to fibers of $p_X := p_Y \circ \pi$. Since ω_X is closed, this subbundle is integrable, hence it defines an integrable connection ∇_S for p_X. We see that $(X \xrightarrow{\pi} Y, \omega, \nabla_S)$ is S-Lagrangian triple.

1.1.6 Proposition. *This correspondence (S-Hamiltonian data on Y) $\to$ (S-Lagrangian triples with given $p_Y : Y \to S$) is bijective.*

Proof: Let us define the inverse correspondence. Let $(X \xrightarrow{\pi} Y, \omega, \nabla_S)$ be an S-Lagrangian triple. The connection ∇_S extends $\omega \in \Omega^2_{X/S}$ to a 2-form $\omega_X \in \Omega^2_X$: one has $\omega_{X|X_s} = \omega_s$ and for $x \in X$ the kernel of $\omega_{X^x} \in \Lambda^2 T^*_{X^x}$ coincides with ∇_S-horizontal vectors at x. Since ∇_S is integrable ω_X is a closed form. Let $(\mathcal{F}_{\omega_X}, \mathrm{curv}_{\omega_X})$ be the corresponding $\Omega^{\geq 1}_X$-torsor, so $\mathcal{F}_{\omega_X} = \Omega^1_X$, $\mathrm{curv}_{\omega_X}(\nu) = d\nu + \omega_X$ (see A1.7). The π-vertical part of zero section of $\mathcal{F}_{\omega_X}$ is a π-descent data for $\mathcal{F}_{\omega_X}$ (see A3.1) which defines $\Omega^{\geq 1}_Y$-torsor $(\mathcal{F}_Y, \mathrm{curv}_Y)$. Recall that a section of $\mathcal{F}_Y$ is a form $\nu \in \pi_* \Omega^1_X$ such that the restriction of ν to fibers of π vanishes and $\mathrm{curv}_Y(\nu) := d\nu + \omega_X \in \Omega^2_Y \subset \pi_* \Omega^2_X$. Denote by $\mathcal{F}_{Y/S}$ the $\Omega^1_{Y/S}$-torsor of sections of $\pi : X \to Y$. One has a canonical isomorphism $r : \mathcal{F}_Y \bmod p_Y^* \Omega^1_S \underset{\sim}{\to} \mathcal{F}_{Y/S}$: here for $\nu \in \mathcal{F}_Y$ $r(\nu)$ is a unique section of π such that $r(\nu)^*(\nu) \in p_Y^* \Omega^1_S \subset \Omega^1_Y$. Let $h : \mathcal{F}_{Y/S} \to \mathcal{F}_Y$ be the map that assigns to a section α of π a unique form $h(\alpha) \in \mathcal{F}_Y$ such that $a^*(h(\alpha)) \in \Omega^1_Y$ vanishes (one has $\alpha = \beta - \alpha^*(\beta)$ for any $\beta \in \mathcal{F}_Y$). Clearly $r \circ h = id_{\mathcal{F}_{Y/S}}$. Let $\widetilde{X} \xrightarrow{\tilde{\pi}} Y$, $\omega_{\widetilde{X}}$, be a twisted cotangent bundle defined by $(\mathcal{F}_Y, \mathrm{curv}_Y)$, so we have the projection $r : \widetilde{X} \to X$ and the section $h : X \to \widetilde{X}$ of r. This is the desired S-Hamiltonian datum on Y.

Remark: The map $\tilde{r} : \widetilde{X} \to X \times_S T^*S$, $\tilde{r}(\tilde{x}) = (r(\tilde{x}), \tilde{x} - h, r(\tilde{x}))$ is isomorphism of symplectic manifolds: Here the symplectic form on $X \times_S T^*S$ is equal to the sum of ω_X and a standard symplectic form on T^*S. $\square$

Consider an S-Hamiltonian datum $(\widetilde{X}, \omega_{\widetilde{X}}, \tilde{\pi}, h)$ on Y. Let $x \in X$ be a point, $y = \pi(x)$, $s = p_X(x)$ be the projections of x. Let $\{t_a\}$ be a local coordinate at s, and q_i be functions at y such that $\{q_i, t_a\}$ are local coordinates at y. Choose a function h_a, p_i at $h(x) \in \widetilde{X}$ such that

$\omega_{\widetilde{X}} = \sum dp_i \wedge dq_i + \sum dh_a \wedge dt_a$. Then $\{q_i, p_i, t_a\}$ are coordinates at x on X, and the Hamiltonian h is given by the functions $h_a(p, q, t)$.

1.1.7 Lemma. *One has* $\nabla_S(\partial_{t_a}) = \partial_{t_a} + \sum_i \partial_{q_i}(h_a)\partial_{p_i} - \partial_{p_i}(h_a)\partial_{q_i}$.

Proof: Follows from $\omega_X(\partial_{p_i} \wedge \nabla_S(\partial_{t_a})) = \omega_X(\partial_{q_i} \wedge \nabla_S(\partial_{t_a})) = 0$. Note that the integrability axiom asserts that $\omega_X(\partial_{t_b} \wedge \nabla_S(\partial_{t_a})) = \partial_{t_b}(h_a) - \partial_{t_a}(h_b) + \sum_i \partial_{p_i}(h_a)\partial_{q_i}(h_b) - \partial_{q_i}(h_a)\partial_{p_i}(h_b) = 0$. □

1.1.8 Corollary. *Let m be a minimal order (with respect to $y \in Y$) of polynomial maps $h_y : X_y \to \widetilde{X}_y$ (note that X_y, $\widetilde{X}_y$ are affine spaces). Then $m - 1$ is equal to the order of corresponding Lagrangian triple (see 1.1.4).* □

1.1.9 Remark: (i) We see that a Hamiltonian datum is just a system of commuting Hamiltonians in a classical sense.

(ii) Let $\mathcal{F}_X$ be the $\Omega^1_{Y/S}$-torsor of sections of $\pi : X \to Y$; one has the map curve$_{\widetilde{X}} \circ h : \mathcal{F}_X \to \Omega^2_Y$, $\mathrm{curv}_{\widetilde{X}} \circ h(\gamma) = (h \circ \gamma)^*(\omega_{\widetilde{X}})$. The equation $\mathrm{curv}_{\widetilde{X}} \circ h(?) = 0$ is a classical Hamilton-Jacobi equation. □

1.2 D-Connections

Let $p : Y \to S$ be any smooth morphism of smooth varieties, and let D_Y be a tdo on Y. Denote by $D_{Y/S}$ the centralizer of $\pi^{-1}\mathcal{O}_S$ in D_Y. This is a flat $\pi^{-1}\mathcal{O}_S$-algebra. One may consider $D_{Y/S}$ as a family of tdo parameterized by S. Namely, for $s \in S$ denote by $\mathfrak{m}_s \subset \mathcal{O}_S$ the maximal ideal of functions equal to zero at s. Then the quotient $D_{Y/S}/\mathfrak{m}_s D_{Y/S}$ is tdo on $Y_s = p^{-1}(s)$ that coincides with the inverse image of D_Y on Y_s (see A3). If $D_Y = D_{\mathcal{L}}$ for some line bundle $\mathcal{L}$, then $D_{Y/S}$ consists of differential operators on $\mathcal{L}$ acting along fibers of p.

1.2.1 Definition. (i) *A D_Y-connection on p is an $\mathcal{O}_S$-linear mapping ∇_{D_Y} : $\mathcal{T}_S \to p_*\mathrm{Der}(D_{Y/S})$ such that for $\tau \in \mathcal{T}_S$, $f \in \mathcal{O}_S$ one has $\nabla_{D_Y}(\tau)(\pi^{-1}f) = \pi^{-1}\tau(f) \subset \pi^{-1}\mathcal{O}_S \subset D_{Y/S}$. Such ∇_{D_Y} is integrable if it commutes with brackets.*

(ii) *A D_Y-connection ∇_{D_Y} is admissible if for any $\tau \in \mathcal{T}_S$ there exists (locally on S) an element $\tilde{\tau} \in p_*D_Y$ such that for any $\partial \in D_{Y/S}$ one has $\nabla_{D_Y}(\tau)(\partial) = [\tilde{\tau}, \partial]$.* $\square$

1.2.2 Remark: One may easily define an obstruction for ∇_{D_Y} to be admissible; it lies in $H^0(S, \Omega^1_S \otimes \mathcal{H}^1_{DR}(Y/S))$. In particular, if the first de Rham cohomology of fibers vanish, any D_Y-connection is admissible. $\square$

We define the order of a D_Y-connection as a smallest n such that $\nabla_{D_Y}(\tau)(\mathcal{O}_Y) \subset D_{Y/S^n} = (D_{Y/S})_n$ for each $\tau \in \mathcal{T}_S$.

1.2.3 Lemma. (i) *One has $\nabla_{D_Y}(\tau)(D_{Y/S^i}) \subset D_{Y/S^{i+n}}$ for any i. Hence we have an $\mathcal{O}_S$-linear map* $\mathrm{gr}\ \nabla_{D_Y} : \mathcal{T}_S \to \mathrm{Der}^{(n)}\mathrm{gr}\ D_{Y/S} = \mathrm{Der}^{(n)} S^{\cdot}\mathcal{T}_{Y/S}$.

(ii) *If $n \geq 1$ then there is a unique $\mathcal{O}_S$-linear map $\sigma\nabla_{D_Y} : \mathcal{T}_S \to p_*S^{n+1}\mathcal{T}_{Y/S}$ such that $(\mathrm{gr}\ \nabla_{D_Y}(\tau))(f) = \{\sigma\nabla_{D_Y}(\tau), f\}$ for $f \in S^{\cdot}\mathcal{T}_{Y/S}$. If ∇_{D_Y} is integrable then the functions $\sigma\nabla_{D_Y}(\tau)$, $\tau \in \mathcal{T}_S$, Poisson commute.*

Proof: (i): Induction by i using $D_{Y/S^i} = \{\partial \in D_{Y/S} : [\partial, \mathcal{O}_Y] \subset D_{Y/S^{i-1}}\}$.

(ii) follows since $\mathrm{gr}\ \nabla_{D_Y}(\tau)$ is a differentiation for Poisson brackets. $\square$.

1.2.4. Now let D_S be a tdo on S. A p-morphism $\alpha : D_S \to D_Y$ is a morphism of $\mathbb{C}$-algebras $\alpha : D_S \to p_*D_Y$ that coincides on $\mathcal{O}_S \subset D_S$ with

$\mathcal{O}_S \xrightarrow{p^{-1}} p_*\mathcal{O}_X \subset p_*D_Y$. Clearly α is injective, so α identifies D_S with a subalgebra in p_*D_Y containing $\mathcal{O}_S$.

1.2.5 Remark: Consider a filtration $L.$ on D_Y by "degree along S": so $L_o = D_{Y/S}$, $L_i = \{\partial \in D_Y : \mathrm{ad}_\partial(\pi^{-1}\mathcal{O}_S) \subset L_{i-1}\}$. One has $\mathrm{gr}^{L\cdot} D_Y = S^\cdot\mathcal{T}_S \otimes_{\mathcal{O}_S} D_{Y/S}$. Then for a p-morphism α one has $\alpha(D_{S^i}) \subset L_i$, and gr α coincides with an obvious embedding $S^\cdot\mathcal{T}_S \hookrightarrow S^\cdot\mathcal{T}_S \otimes_{\mathcal{O}_S} D_{Y/S}$.

1.2.6. Let $\alpha : D_S \to p_*D_Y$ be a p-morphism. One associates with α an admissible integrable D_Y-connection on p as follows. For $\tau \in \mathcal{T}_S$ choose $\tilde{\tau} \in \widetilde{\mathcal{T}}_{D_S}$ such that $\sigma(\tilde{\tau}) = \tau$. Then $\alpha(\tilde{\tau}) \in L_1$, hence $\mathrm{ad}_\alpha(\tilde{\tau})$ maps $D_{Y/S}$ to itself. Put $\nabla_\alpha(\tau) := \mathrm{ad}_{\alpha(\tau)}|_{D_{Y/S}} \in \mathrm{Der}\ D_{Y/S}$. It is easy to see that $\nabla_\alpha(\tau)$ does not depend on choice of $\tilde{\tau}$. This morphism $\nabla_\alpha : \mathcal{T}_S \to \mathrm{Der}\ D_{Y/S}$ is our D_Y-connection. It is admissible and integrable.

1.2.7 Lemma. *If the fibers of p are connected, then $(D_S, \alpha) \mapsto \nabla_\alpha$ is a bijection between the set of pairs (D_S, α) and admissible integrable D_Y-connections on p.*

Proof: Here is a construction of an inverse map. For an admissible integrable connection $\nabla = \nabla_{D_Y}$ put $\widetilde{\mathcal{T}}_\nabla = \{(\tau, \tilde{\tau}) \in \mathcal{T}_S \times p_*D_Y$: for any $\partial \in D_{Y/S}$ one has $\nabla(\tau)(\partial) = [\tilde{\tau}, \partial]\}$. One has short exact sequence $0 \to \mathcal{O}_S \xrightarrow{i} \widetilde{\mathcal{T}}_\nabla \xrightarrow{\sigma} \mathcal{T}_S \to 0$, where $i(f) = (0, p^{-1}(f))$, $\sigma(\tau, \tilde{\tau}) = \tau$, and an obvious $\mathcal{O}_S$-module and Lie algebra structure on $\widetilde{\mathcal{T}}_\nabla$ make $\widetilde{\mathcal{T}}_\nabla$ an $\mathcal{O}_S$-extension of $\mathcal{T}_S$ (see A1.3, A1.4). Let D_S^∇ be the corresponding tdo. The embedding $\widetilde{\mathcal{T}}_\nabla \to p_*D_Y$, $(\tau, \tilde{\tau}) \mapsto \tilde{\tau}$, extends uniquely to a morphism of rings $\alpha_\nabla : D_S^\nabla \to p_*D_Y$ which is a p-morphism. This $(D_S^\nabla, \alpha_\nabla)$ is a desired pair. $\square$

1.2.8 Remark: For a p-morphism $\alpha : D_S \to D_Y$ consider the smallest integer m such that $\alpha(\widetilde{T}_{D_S}) \subset D_{Y_m}$. Then $m-1$ is equal to the order of ∇_α, and $\sigma\nabla_\alpha(\tau) = \alpha(\tilde{\tau}) \bmod D_{Y_{m-1}} \in S^m\mathcal{T}_{Y/S}$ for $\tilde{\tau} \in \widetilde{\mathcal{T}}_{D_S}$, $\tau = \sigma\tilde{\tau} \in \mathcal{T}_S$.

1.3 Quantization

Let $(X \xrightarrow{\pi} Y \xrightarrow{p_Y} S; \omega; \nabla_S)$ be an S-Lagrangian triple (see 1.1.3), so we have a filtered commutative $\mathcal{O}_Y$-algebra $A = \pi_*\mathcal{O}_X$ with Poisson bracket $\{\ ,\ \}$, and the $\Omega_Y^{\geq 1}$-torsor $(\mathcal{F}_Y, \mathrm{curv}_Y)$ (see 1.1.6: this torsor corresponds to the twisted cotangent bundle of the Hamiltonian datum). Let $\Omega = \det\ \Omega^1_{Y/S}$ be the sheaf of volume forms along the fibers of p_Y, and $(\mathcal{F}_\Omega, \mathrm{curv}_\Omega) = d\log\Omega$ be the corresponding $\Omega_Y^{\geq 1}$-torsor (see A1.12). Put $(\mathcal{F}_Y^\wedge, \mathrm{curv}_Y^\wedge) = (\mathcal{F}_Y, \mathrm{curv}_Y) + \frac{1}{2}(\mathcal{F}_\Omega, \mathrm{curv}_\Omega)$; let $D_Y = D_{(\mathcal{F}_Y^\wedge, \mathrm{curv}_Y^\wedge)}$ be the corresponding tdo.

¿From now on we will assume that our Lagrangian triple has order 1, i.e., for $\tau \in \mathcal{T}_S$ one has $\nabla_S(\tau)(\mathcal{O}_Y) \subset A_1$. According to A2.5 one has a canonical isomorphism $\tilde{\sigma} : \widetilde{\mathcal{T}}_{D_{Y/S}} = D_{Y/S} \xrightarrow{\sim} A_1$.

1.3.1 Definition. *A quantization of our Lagrangian triple is an order 1 integrable D_Y-connection ∇_{D_Y} on p_Y such that for $\tau \in \mathcal{T}_S$, $f \in \mathcal{O}_Y = D_{Y/S^0}$ one has $\tilde{\sigma}[\nabla_{D_Y}(\tau)(f)] = \nabla_S(\tau)(f)$.* □

Let ∇_{D_Y} be any order 1 integrable connection. The following lemma explains how to verify whether ∇_{D_Y} is a quantization, and also why we took the $\Omega^{1/2}$-twist in the definition of D_Y. Consider the following sheaves on Y:

$\mathcal{F}^A := \{(\tau, \ell), \tau \in p_Y^{-1}\mathcal{T}_S, \ell : \mathcal{O}_Y \to A_1 | \ell(fg) = f\ell(g) + g\ell(f), \{\ell(f), g\} = \{\ell(g), f\}$ for $f, g \in \mathcal{O}_Y$, $\ell(t) = \tau(t)$ for $t \in p_Y^{-1}\mathcal{O}_S \subset \mathcal{O}_Y\}$.

$\mathcal{F}^D := \{(\tau,\ell'), \tau \in p_Y^{-1}\mathcal{T}_S, \ell' : \mathcal{O}_Y \to A_1 | \ell'(fg) = f\ell'(g)+\ell'(f)g, [\ell'(f),g]+[f\ell'(g)] = 0$ for $f,g \in \mathcal{O}_Y$, $\ell'(t) = \tau(t)$ for $t \in p_Y^{-1}\mathcal{O}_S \subset \mathcal{O}_Y\}$.

One has a short exact sequence of $p_Y^{-1}\mathcal{O}_S$-modules:

$$0 \to A_2/A_0 \xrightarrow{i_A} \mathcal{F}^A \xrightarrow{j_A} p_Y^{-1}\mathcal{T}_S \to 0, \quad 0 \to D_{Y^2}/D_{Y^0} \xrightarrow{i_D} \mathcal{F}^D \xrightarrow{j_D} p_Y^{-1}\mathcal{T}_S \to 0,$$

defined by formulas $i_A(a) = (0,\ell(a))$, $\ell(a)(f) = \{a,f\}$, $j_A(\tau,\ell) = \tau$, $i_D(\partial) = (0,\ell'(\partial))$, $\ell'(\partial)(f) = [\partial,f]$, $j_D(\tau,\ell') = \tau$. Our connections ∇_S, ∇_{D_Y} define the splittings ∇_S^0, ∇_D^0 of j_A, j_D, respectively, by formulas $\nabla_S^0(\tau) = (\tau, \nabla_S(\tau)|_{\mathcal{O}_Y})$, $\nabla_D^0(\tau) = (\tau, \nabla_{D_Y}(\tau)|_{\mathcal{O}_Y})$.

1.3.2 Lemma. (i) *One has a canonical commutative diagram*

$$\begin{array}{ccccccccc}
0 & \to & D_{Y_2}/D_{Y_0} & \to & \mathcal{F}^D & \to & p_Y^{-1}\mathcal{T}_S & \to & 0 \\
 & & \wr\downarrow \tilde{\sigma} & & \wr\downarrow \tilde{\sigma}_{\mathcal{F}} & & || & & \\
0 & \to & A_2/A_0 & \to & \mathcal{F}^A & \to & p_Y^{-1}\mathcal{T}_S & \to & 0
\end{array}$$

where $\tilde{\sigma}_{\mathcal{F}}$ is defined by formula $\tilde{\sigma}_{\mathcal{F}}(\tau,\ell') = (\tau,\ell)$, $\ell(f) = \tilde{\sigma}\ell'(f)$, and $\tilde{\sigma} : D_{Y_i}/D_{Y_{i-2}} \underset{\sim}{\to} A_i/A_{i-2}$ was defined in A2.5.

(ii) ∇_{D_Y} *is a quantization iff $\tilde{\sigma}_{\mathcal{F}}\nabla_D^0 - \nabla_S^0 \in \mathrm{Hom}(p_Y^{-1}\mathcal{T}_S, A_2/A_0)$ is 0. In particular, ∇_{D_Y} is always a quantization if $p_{Y_*}(A_2/A_0) = 0$.*

Proof: (i) It suffices to verify that $\tilde{\sigma}_{\mathcal{F}}(\tau,\ell')$ actually lies in $\mathcal{F}^A$ by a direct computation.

(ii) Clear. □

Let ∇_{D_Y} be a quantization.

1.3.3. Lemma. (i) *One has $\sigma(\nabla_S) = \sigma(\nabla_{D_Y}) \in \Omega_S^1 \otimes p_{Y_*}S^2\mathcal{T}_{Y/S}$.*

(ii) ∇_{D_Y} is admissible D_Y-connection.

Proof: (i) Clear, (ii) follows since π has affine fibers, see 1.2.1. $\square$

According to 1.2.7 a quantization ∇_{D_Y} defines a tdo D_S on S together with embedding $\alpha : D_S \hookrightarrow p_{Y*}D_Y$, which is our primary object of interest.

1.4 Symmetries

Assume that we are in a situation 1.2, i.e., we have a smooth map $p_Y : Y \to S$ and a tdo D_Y on Y. Let $\nu_Y : \mathfrak{g} \to \mathcal{T}_Y$, $\nu_S : \mathfrak{g} \to \mathcal{T}_S$ be actions of a Lie algebra $\mathfrak{g}$ on Y and S that commute with p. Let ν_{D_Y}Der D_Y be a weak ν_Y-action of $\mathfrak{g}$ on a tdo D_Y (see A4.1). Clearly the derivations $\nu_{D_Y}(\gamma)$, $\gamma \in \mathfrak{g}$, preserve the subalgebra $D_{Y/S} \subset D_Y$.

1.4.1 Definition. (i) *The action ν_{D_Y} preserves a D_Y-connection ∇_{D_Y} for p if for any $\gamma \in \mathfrak{g}$, $\tau \in \mathcal{T}_S$ one has $[\nu_{D_Y}(\gamma), \nabla_{D_Y}(\tau)] = \nabla_{D_Y}([\nu_S(\gamma), \tau]) \in$ Der $D_{Y/S}$.*

(ii) *The action ν_{D_Y} preserves a p-morphism $\alpha : D_S \to D_Y$ if the derivations $\nu_{D_Y}(\gamma)$, $\gamma \in \mathfrak{g}$, preserve a subalgebra $D_S \overset{\alpha}{\hookrightarrow} p_*D_Y$.* $\square$

It is easy to see that if ν_{D_Y} preserves α, then it preserves ∇_α (see 1.2.4); conversely if the fibers of p are connected, then ν_{D_Y} preserves α_∇ if it preserves ∇_{D_Y} (see 1.2.5).

Assume that ν_{D_Y} preserves a p-morphism α. Then the restriction of operators $\nu_{D_Y}(\gamma)$, $\gamma \in \mathfrak{g}$, to $D_S \overset{\alpha}{\hookrightarrow} p_*D_Y$ define a weak ν_S-action ν_{D_S} of $\mathfrak{g}$ on D_S.

1.4.2. Let $(X \overset{\pi}{\to} Y; \omega; \nabla_S)$ be an S-Lagrangian triple, and our Lie algebra $\mathfrak{g}$ acts on it. This means that we have compatible $\mathfrak{g}$-actions ν_X, ν_Y, ν_S on

X, Y and S that fix ω and ∇_S (note that, since π and p_Y are surjective, ν_Y and ν_S are uniquely determined by ν_X). We get a canonical weak ν_X-action on D_{ω_X} and weak ν_Y-action of $\mathfrak{g}$ on D_Y (since D_{ω_X}, D_Y were defined in a canonical way). We will say that $\mathfrak{g}$ preserves a quantization if ν_{D_Y} preserves ∇_{D_Y}. In this case we get a weak ν_S-action of $\mathfrak{g}$ on corresponding D_S.

Sometimes one needs strong actions on D_{ω_X}, D_Y rather than just weak ones. One has

1.4.3 Lemma. *The strong liftings $\tilde{\nu}_{\omega_X} : \mathfrak{g} \to \widetilde{\mathcal{T}}_{D_{\omega_X}}$ for ν_{ω_X} are in 1–1 correspondence with ones $\tilde{\nu}_{D_Y} : \mathfrak{g} \to \widetilde{\mathcal{T}}_{D_Y}$ for ν_{D_Y}.*

Proof: Let $N \subset \widetilde{\mathcal{T}}_{D_{\omega_X}}$ be a normalizer of $\nabla_{\omega_X}(\mathcal{T}_{X/Y})$. One has $\widetilde{\mathcal{T}}_{D_Y} = \pi_*(N/\nabla_{\omega_X}(\mathcal{T}_{X/Y}))$. Now assume we have $\tilde{\nu}_{\omega_X}$. Clearly $\tilde{\nu}_{\omega_X}(\mathfrak{g}) \subset N$, hence $\tilde{\nu}_{D_Y} := \tilde{\nu}_{\omega_X} \mod \nabla_{\omega_X}(\mathcal{T}_{X/Y})$ is a strong lifting of $\tilde{\nu}_{D_Y}$. Conversely, assume we have $\tilde{\nu}_{D_Y}$. For $\gamma \in \mathfrak{g}$ an element $\tilde{\nu}_{\omega_X}(\gamma) \in N$ such that $\mathrm{ad}_{\tilde{\nu}_{\omega_X}}(\gamma) = \nu_{\omega_X}(\gamma)$ defines it up to a constant. The condition that $\tilde{\nu}_{\omega_X}(\gamma) \mod \nabla_{\omega_X}\mathcal{T}_{X/Y}) = \tilde{\nu}_{D_Y}$ defines it uniquely. □

Note that $\tilde{\nu}_{\omega_X}$ is just an ω_X-Hamiltonian lifting of ν_{ω_X} (see A4.3(ii)).

1.5 Kostant D-modules

Assume we are in a situation 1.2, so we have $p : Y \to S$, a tdo D_Y on Y, D_S on S and a p-morphism $\alpha : D_S \to D_Y$. Let M be a D_Y-module. The algebra p_*D_Y acts on sheaf-theoretic direct images R^ip_*M in an obvious manner, hence α defines the functors $R^ip_* : D_Y$-modules $\to D_S$-modules. If R^ip_* transforms $\mathcal{O}_Y$-coherent modules to $\mathcal{O}_S$-coherent ones, then it transforms lisse D_Y-modules to lisse D_S-ones (see A1.14).

1.5.1. Assume we have an action of a Lie algebra $\mathfrak{g}$ on our data such that ν_{D_Y} preserves α (see 1.4.1). Let $\tilde{\nu}_{D_Y} : \mathfrak{g} \to \widetilde{\mathcal{T}}_{D_Y}$, $\tilde{\nu}_{D_S} : \mathfrak{g} \to \widetilde{\mathcal{T}}_{D_S}$ be strong liftings of ν_{D_Y}, ν_{D_S}. For a D_Y-module M consider a canonical ν_{D_Y}-action ν_M^0 of $\mathfrak{g}$ on M (see A4.4). The induced action of $\mathfrak{g}$ on R^ip_*M is obviously a ν_{D_S}-action. Hence $\tilde{\nu}_{D_S}$ defines a canonical action $[\nu_M^0] : \mathfrak{g} \to \mathrm{End}_{D_S} R^ip_*M$ of $\mathfrak{g}$ on R^ip_*M (see A4.5). We get a canonical action of $\mathfrak{g}$ on the functor R^ip_*, i.e., R^ip_* transforms D_Y-modules to $D_S \otimes_{\mathbb{C}} U(\mathfrak{g})$-ones.

Now let $(X \overset{\pi}{\to} Y; \omega; \nabla_S)$ be an S-Lagrangian triple.

1.5.2 Definition. (i) *Kostant line bundle is a line bundle $\mathcal{L}_Y$ on Y equipped with a D_Y-module structure (which is an isomorphism $D_Y \overset{\sim}{\to} D_{\mathcal{L}_Y}$).*

(ii) *An ω_X-line bundle is a line bundle $\mathcal{L}_X$ on X equipped with a D_{ω_X}-module structure (which is the same as a connection ∇_X on $\mathcal{L}_X$ with* $\mathrm{curv}\, \nabla_X = \omega_X$*).*

(iii) *An ω_X-line bundle $(\mathcal{L}_X, \nabla_X)$ is admissible if for any $y \in Y$ its restriction of $(\mathcal{L}_{X_y}, \nabla_{X_y})$ to the fiber X_y is a trivial bundle with connection.* □

1.5.3 Remark: Since the fibers X_y are affine spaces, in analytic situation any ω_X-line bundle is admissible. In algebraic situation admissibility just means that ∇_{X_y} has regular singularities at infinity (see [Bo], [D]). □

Assume that there exists a line bundle $\Omega^{1/2}$ on Y together with an isomorphism $(\Omega^{1/2})^{\otimes 2} \overset{\sim}{\to} \Omega$ (for notations see 1.3); choose one. Let M be a D_Y-module. Then $M_{\Omega^{-1/2}} := \Omega^{-1/2} \otimes_{\mathcal{O}_Y} M$ is a $D_{(\mathcal{F}_Y, \mathrm{curv}_Y)}$-module. Since $D_{(\mathcal{F}_Y, \mathrm{curv}_Y)}$ coincides with π-descent of D_{ω_X}, we see that $\pi^* M_{\Omega^{-1/2}} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} M_{\Omega^{-1/2}}$ is a D_{ω_X}-module. If M is a line bundle, then $\pi^* M_{\Omega^{-1/2}}$ is an admissible ω_X-bundle, so we obtained the functor $\pi^*_{\Omega^{1/2}}$: (Kostant line bundles) $\to$ (admissible ω_X-bundles), $\pi^*_{\Omega^{1/2}}(\mathcal{L}_Y) = \pi^*(\Omega^{-1/2} \otimes \mathcal{L}_Y)$.

1.5.4 Lemma. *This functor is equivalence of categories.*

Proof: The inverse functor assigns to $(\mathcal{L}_X, \nabla_X)$ a line bundle $\Omega^{1/2} \otimes \pi_* \mathcal{L}_X^{\nabla_{X/Y}}$, where $\nabla_{X/Y}$ is "vertical" part of ∇_X. $\square$

1.5.5. Let ∇_{D_Y} be a quantization of our symplectic triple, and D_S be the corresponding tdo on S. Let $\mathcal{L}_Y$ be a Kostant line bundle. Then $R^i p_{Y_*} \mathcal{L}_Y$ are D_S-modules, we will call them *Kostant D_S-modules.* If a Kostant D_S-module $\mathcal{E}$ is lisse (which happens, e.g., when p_Y is proper), then $\mathcal{E}$ is a vector bundle on S with a canonical integrable projective connection (see A1.14–A1.17).

1.5.6. Assume we are in a situation 1.4.2, so we have a Lie algebra $\mathfrak{g}$ that acts on our Lagrangian triple and preserves a quantization ∇_{D_Y}. Choose strong liftings $\tilde{\nu}_{D_Y}$, $\tilde{\nu}_{D_S}$ By 1.5.1 these define an action of $\mathfrak{g}$ on Kostant D-module are $\mathfrak{g}$-modules.

1.5.7 Remark: $\tilde{\nu}_{D_Y}$ is the same as ν_{D_Y}-action of $\mathfrak{g}$ on a Kostant line bundle.

1.6 Example: Metaplectic Representation

Let W be a symplectic $\mathbb{C}$-vector space with symplectic form ω. Let S be a Grassmannian of Lagrangian planes in W, and $L \subset W_S$ be a canonical Lagrangian subbundle of a constant vector bundle W_S on S. Denote by $S^\wedge$ the space of the line bundle det L on S with zero section removed.

Define an S-Lagrangian triple $(X \overset{\pi}{\to} Y;\omega;\nabla_S)$ as follows. Put $X = W_S = W \times S$, $Y = W_S/L$, $\pi =$ canonical projection, ω_X is a lifting to X of a constant 2-form ω on W, $\nabla_S =$ constant connection. We may also consider an $S^\wedge$-Lagrangian triple $(X^\wedge \overset{\pi^\wedge}{\to} Y^\wedge;\omega^\wedge;\nabla_{S^\wedge})$ defined in the same way (this is just a base change by $S^\wedge \to S$ of the previous triple).

Let $\mathfrak{g} = W \rtimes Sp(W)$ be a Lie algebra of affine symplectic symmetries of W (so W acts by translations). It acts on our Lagrangian triples in an obvious manner (so W acts trivially on S, $S^\wedge$).

Let $\tilde{\mathfrak{g}}$ be a central extension of $\mathfrak{g}$ by $\mathbb{C}$ such that for $\widetilde{w}_1$, $\widetilde{w}_2 \in \widetilde{W} \subset \tilde{\mathfrak{g}}$ one has $[\widetilde{w}_1, \widetilde{w}_2] = \omega(w_1 \wedge w_2)$. Such $\widetilde{\mathfrak{g}}$ exists and unique up to a unique isomorphism. In fact $H^1(\tilde{\mathfrak{g}}, \mathbb{C}) = H^2(\tilde{\mathfrak{g}}, \mathbb{C}) = 0$. By A4.2 one has a canonical strong lifting $\tilde{\nu}_{D_Y} : \tilde{\mathfrak{g}} \to \widetilde{\mathcal{T}}_{D_Y}$. It restricts to the Lie algebra map $\widetilde{W} \to \widetilde{\mathcal{T}}_{D_{Y/S}}$ which defines the isomorphism of associative $\mathcal{O}_S$-algebra $U_1(\widetilde{W}) \otimes_{\mathbb{C}} \mathcal{O}_S \ \widetilde{\to}\ D_{Y/S}$; here $U_1(\widetilde{W})$ is a quotient of a universal envelopping algebra $U(\widetilde{W})$ modulo relation $1 = 1 \in \mathbb{C} \subset \widetilde{W}$. Let ∇_{D_Y} be a D_Y-connection for p_Y with $U_1(\widetilde{W})$ being the horizontal sections. This is a quantization of our Lagrangian triple; let D_S be the corresponding tdo on S. The $\mathfrak{g}$-action preserves the quantization and, as above, we get a canonical strong lifting $\tilde{\nu}_{D_S} : \widetilde{\mathfrak{g}} \to \widetilde{\mathcal{T}}_{D_S}$. Certainly, in all these things we may replace the S-Lagrangian triple by the $S^\wedge$-one.

It is easy to see that $\Omega^{1/2}$ does not exist globally on Y. On $Y^\wedge$ the sheaf Ω is canonically trivialized, hence we get a canonical $\Omega^{1/2}$. Take an admissible $\omega_{X^\wedge}$-bundle (it comes from a line bundle $\mathcal{L}_W$ on W equipped with a connection ∇_W with curvatuer ω). By 1.5.4 we get a Kostant line bundle $\mathcal{L}_{Y^\wedge}$ on $Y^\wedge$, hence a Kostant $D_{S^\wedge}$-module $\mathcal{E} = p_{Y^\wedge_*}(\mathcal{L}_{Y^\wedge})$ on $S^\wedge$. By

1.5.6 it carries a canonical "metaplectic" $\tilde{\mathfrak{g}}$-action: for $s \in S^\wedge$ a fiber $\mathcal{E}_s$ is a metaplectic representation of $\tilde{\mathfrak{g}}$ on vectors "algebraic with respect to a polarization L_s."

1.7 Complex Polarizations

In this section we will relate the above purely holomorphic construction with a complex polarization approach. We will start with a general lemma on a C^∞-description of twisted cotangent bundles. Everywhere below "variety" means "complex analytic variety."

Let Y be a smooth variety and $\phi = (\pi_\phi : X_\phi \to Y; \omega_\phi)$ be a twisted cotangent bundle over Y. Let $(\mathcal{F}_\phi, \mathrm{curv})$ be the corresponding $\Omega_\gamma^{\geq 1}$-torsor of holomorphic sections of π_ϕ, and let $C^\infty\mathcal{F}_\phi$ be the $\Omega^{10}_{C^\infty Y}$-torsor of C^∞ sections of π_ϕ (so $C^\infty\mathcal{F}_\phi$ is the pushout of $\mathcal{F}_\phi$ by $\Omega^1_Y \to \Omega^{10}_{C^\infty Y}$). For $\gamma \in C^\infty\mathcal{F}_\phi$ put $\mathrm{curv}(\gamma) := \gamma^*(\omega_\phi)$: this is a closed C^∞-class 2-form on Y with zero (0,2)-component.

1.7.1 Lemma. *The map $(\phi, \gamma) \mapsto \mathrm{curv}(\gamma)$ is a 1-1 correspondence between the set of pairs (twisted cotangent bundle ϕ on Y, a C^∞-class section of π_ϕ) and the set of closed C^∞-class 2-forms with zero (0,2)-component.*

Proof: Here is a construction of inverse map. Let $\nu = \nu^{11} + \nu^{20}$ be a closed C^∞-form. We need to construct an $\Omega^{10}_{C^\infty Y}$-trivialized $\Omega_Y^{\geq 1}$-torsor. Since $\bar\partial\nu^{11} = 0$ the sheaf $\mathcal{F}_\nu := \bar\partial^{-1}(-\nu^{11}) \subset \Omega^{10}_{C^\infty Y}$ is an Ω^1_Y-torsor; it carries an obvious $\Omega^{10}_{C^\infty Y}$-trivialization. Define $\mathrm{curv}_\nu : \mathcal{F}_\nu \to \Omega^2_Y$ formula $\mathrm{curv}_\nu(\gamma) = d\gamma + \nu$. This $(\mathcal{F}_\nu, \mathrm{curv}_\nu)$ is our $\Omega_Y^{\geq 1}$-torsor. $\square$.

1.7.2 Remarks: (i) Consider the sheaf $A = \pi_{\phi^*}\mathcal{O}_{X_\phi}$; it carries a canonical filtration A_i (see A1.3). A C^∞-section γ defines the map $\gamma^* : A \to \mathcal{O}_{C^\infty Y}$. If $\mathrm{curv}(\gamma) = \omega_\gamma$ is a nondegenerate 2-form then γ^* is injective and one may determine $A_i \overset{\gamma^*}{\hookrightarrow} \mathcal{O}_{C^\infty Y}$ by induction: one has $A = \mathcal{O}_Y$, $A_i = \{f \in \mathcal{O}_{C^\infty}\} : \{f, \mathcal{O}_Y\} \subset A_{i-1}$; here $\{\ \ \}$ is Poisson bracket on $\mathcal{O}_{C^\infty Y}$ defined by ω_γ.

(ii) Certainly 1.7.1 is a particular case of a general nonsense that claims, in the notations of A1.5, that a quasi-isomorphism $A^{\cdot} \to B^{\cdot}$ of length 2 complexes defines an equivalence between categories of $A^{\cdot}$- and $B^{\cdot}$-torsors.$\square$

Consider an S-Lagrangian triple $(X \xrightarrow{\pi} Y; \omega; \nabla_S)$.

1.7.3 Definition. *A C^∞-class section $\gamma : Y \to X$ is called admissible if it satisfies the properties (i)–(iii) below:*

(i) *∇_S is tangent to $\gamma(Y)$, i.e., for $y \in Y$ the $\mathbb{R}$-subspace $d\gamma(T_{Y,y}) \subset T_{X_{\gamma(y)}}$ contains the ∇_S-horizontal subspace $\nabla_S(T_{S_{p_Y(y)}})_{\gamma(y)}$. Clearly the ∇_S-horizontal planes tangent to $\gamma(Y)$ form an integrable C^∞-class connection ∇_S^γ for p_Y.*

(ii) *This ∇_S^γ is globally trivial, i.e., it comes from a global C^∞-class trivialization $Y \simeq Y_0 \times S$. Consider a C^∞-class 2-form $\omega^\gamma := \gamma^*(\omega)$ along the fibers of p_Y.*

(iii) *For $s \in S$ the form ω_s^γ on Y_s is nondegenerate and real-valued.* $\square$.

1.7.4 Lemma. (i) *The form ω_s^γ is a closed form of type (1,1) on Y_s.*

(ii) *ω^γ is ∇_S^γ-horizontal, i.e., by 1.7.3(ii), it comes from a single symplectic form ω_0 on Y_0.*

(iii) *For each $y_0 \in Y_0$ the section $S \to Y_0 \times S = Y$, $s \mapsto (y_0, s)$, is holomorphic.*

Proof: Clear. $\square$

Let us describe the above structure from a Y_0 viewpoint.

Let (Y_0, ω_0) be any C^∞-class (real) symplectic manifold.

1.7.5 Definition. *A complex polarization of* (Y_0, ω_0) *is a complex structure on* Y_0 *such that* ω_0 *has type (1,1).* $\square$

According to integrability theorem of Newlander-Nirenberg, a complex structure s on Y_0 is the same as an integrable $\mathbb{C}$-subbundle $T_s^{01} \subset T_{Y_0} \otimes \mathbb{C}$ such that $T_s^{01} \oplus \bar{T}^{01} \simeq T_{Y_0} \otimes \mathbb{C}$ (here "integrable" means $[T_s^{01}, T_s^{01}] \subset T_s^{01}$). Such s is a complex polarization iff T_s^{01} is an ω_0-Lagrangian subbundle.

1.7.6. Note that 1-jet of a deformation of a $\mathbb{C}$-subbundle $T_s^{01} \subset T_{Y_0} \otimes \mathbb{C}$ is an element $\varphi \in \mathrm{Hom}(T_s^{01}, T_s^{01}) = \Omega_s^{01} \otimes T_s^{10}$, where$T_s^{10} := T_{Y_0} \otimes \mathbb{C}/T_s^{01}$, $\Omega_S^{01} := (T_s^{01})^*$. If T_s^{01} is a complex structure, then φ is a 1-jet of a deformation of complex structure iff $\bar{\partial}\varphi \in \Omega_s^{02} \otimes T_s^{10}$ is equal to zero (here $\bar{\partial}$ is taken with respect to the holomorphic structure on T_s^{10}). If T_s^{01} is Lagrangian, then ω_0 identifies Ω_s^{01} with T_s^{10}, and φ is a 1-jet of a deformation of a Lagrangian subbundle iff $\varphi \in S^2 T_s^{10} \subset T_s^{10} \otimes T_s^{10}$. If T_s^{01} is a complex polarization and both above-mentioned conditions hold, then φ is a 1-jet of a deformation of a polarization.

Let S be a C^∞ manifold, and T_s^{01}, $s \in \mathbb{C}$, be a C^∞-class family of complex polarizations of (Y_0, ω_0). Put $Y := Y_0 \times S$. Our T_s^{01} form a subbundle $T_{Y/S}^{01}$ of $T_{Y/S} \otimes \mathbb{C}$, same for $T_{Y/S}^{01}$, etc. The 1-jets of deformation form a section $C \in \Omega^1_{C^\infty S} \otimes S^2 T_{Y/S}^{10}$.

Assume now that S is a $\mathbb{C}$-analytic manifold.

1.7.7 Definition. *A S family of polarizations is holomorphic if* $C \in \Omega^1_S \otimes_{\mathcal{O}_S} S^2 T^{10}_{Y/S} = \Omega^{10}_{C^\infty S} \otimes S^2 T^{10}_{Y/S}$.

1.7.8. Proposition. *One has a canonical 1–1 correspondence between S-Lagrangian triples $(X \xrightarrow{\pi} Y; \omega; \nabla_S)$ equipped with an admissible C^∞ section $\gamma : Y \to X$, and a C^∞-class (real) symplectic manifolds (Y_0, ω_0) equipped with a holomorphic S family of polarizations.*

Proof: As was explained in 1.7.3, 1.7.4 an admissible section defines (Y_0, ω_0) and a holomorphic S-family of polarizations. Conversely, consider a holomorphic family of polarizations of (Y_0, ω_0). Put $Y = Y_0 \times S$. The subbundle $T^{01}_Y \subset T_Y \otimes \mathbb{C}$ with fiber at $(y, s) \in Y$ equal to $T^{01}_{Y_s}(y) \oplus T^{01}_S(s)$ defines the complex structure on Y such that the projection $p_Y : Y \to S$ is holomorphic. Let ω_Y be the inverse image of ω_0 via the projection $Y \to Y_0$. This is a closed (1,1)-form on Y. Let $(\widetilde{X}, \omega_{\widetilde{X}})$, $\tilde{\pi} : \widetilde{X} \to Y$ be the twisted cotangent bundle over Y with the C^∞-section $\tilde{\gamma} : Y \to \widetilde{X}$ defined by ω_Y according to 1.7.1. Put $X = \widetilde{X} \bmod p_Y^* \Omega^1_S \xrightarrow{\pi} Y$: this is a twisted cotangent bundle along the fibers of p_Y. By 1.7.1 a holomorphic section of X is a C^∞-class 1-form ν along the fibers of π (which is the same as a family ν_s of 1-forms on Y_0 parameterized by $s \in S$) such that ν_s is a 10-form on Y_S (i.e., $\nu_s|_{T^{01}_s} = 0$), $\bar{\partial}\nu_s = \omega_0 \in \Omega^{11}_{Y_s}$ and ν_s depends on $s \in S$ in a holomorphic way. Denote by $H(\nu)$ the 1,0-form on Y which coincides with ν in fiberwise directions and vanishes on horizontal ones (i.e., $H(\nu)|_{y_o \times S} = 0$ for each $y_0 \in Y_0$). One has $\bar{\partial}\nu = \omega_Y$, hence we have defined a holomorphic section $H : X \to \widetilde{X}$. This $(\widetilde{X}, H)$ is an S-Hamiltonian datum on Y, so, by 1.1.6, we have S-Lagrangian triple $(X \xrightarrow{\pi} Y, \omega; \nabla_S)$. It is easy to see that $\gamma = \tilde{\gamma} \bmod p_Y^* \Omega^1_S$ is an admissible section. This construction is clearly inverse to one of 1.7.3, 1.7.4.

□

1.7.9 Lemma. *Consider a holomorphic S-family of polarizations of* (Y_0, ω_0). *The corresponding S-Lagrangian triple has order $\leq n$ (see 1.1) iff for any $s \in S$ and a tangent vector ∂_s at s the tensor $C(s) \in S^2 T_s^{10}$ (see 1.7.7) lies in $A_{n-1} \cdot S^2 T_{Y_s}$. Here $A_{n-1} = A_{sn-1} \subset \mathcal{O}_{C^\infty Y_0}$ is the sheaf of functions on Y_0 defined in 1.7.2(i) for the complex structure Y_s and the form ω_0. For example, our triple has order 1 iff $C(s)$ is a holomorphic tensor on Y_s.*

Proof: Clear. □

2. D-Rational Varieties and Canonical Quantization

In some situations a quantization is uniquely defined by a Lagrangian triple. In this section we desribe some sufficient conditions for this.

2.1 D-Rationality

Let Y be a smooth variety and D be a tdo on Y.

2.1.1 Definition. *Y is D-rational if $H^0(Y,D)=\mathbb{C}$ and $H^i(Y,D)=0$ for $i>0$.* $\square$

For arbitrary D consider the class $c_1'(D):=c_1(D)-\frac{1}{2}c_1(\det\Omega_Y^1)\in H^1(Y,\Omega_Y^{\geq 1})$. For $c\in H^1(Y,\Omega_Y^{\geq 1})$ let $\bar{c}$ denote the image of c in $H^1(Y,\Omega_Y^1)$. Let $\delta_D: H^j(Y,S^i\mathcal{T}_Y)\to H^{j+1}(Y,S^{i-1}\mathcal{T}_Y)$ be the convolution with $\bar{c}_1'(D)$.

2.1.2 Lemma. *Assume that $H^0(Y,\mathcal{O}_Y)=\mathbb{C}$ and for each $i>0$ the sequence*

$$0\to H^0(Y,S^i\mathcal{T}_Y)\stackrel{\delta_D}{\to}H^1(Y,S^{i-1}\mathcal{T}_Y)\stackrel{\delta_D}{\to}\cdots\stackrel{\delta_D}{\to}H^i(Y,\mathcal{O}_Y)\to 0$$

is exact. Then Y is D-rational.

Proof: By A2.6 δ_D is the boundary map for the short exact sequence $0\to S^{i-1}\mathcal{T}_Y\to D_i/D_{i-2}\to S^i\mathcal{T}_Y\to 0$, i.e., δ_D is the first differential in the spectral sequence $E^{p,q}$ that computes $H^{\cdot}(Y,D)$ using filtration D_i. Our conditions mean that $E_2^{0,0}=\mathbb{C}$, $E_2^{p,q}=0$ for $p,q\neq(0,0)$. $\square$

2.1.3 Remark: One may interpret δ_D microlocally as follows. Let $\pi: T^*Y\to Y$ be cotangent bundle to Y. The symplectic form on T^*Y defines the isomorphism $\Omega^1_{T^*Y} \underset{\sim}{\to} \mathcal{T}_{T^*Y}$ (which coincides with translation action along the fibers on $\pi^*\Omega^1_Y \subset \Omega^1_{T^*Y}$). Hence we get the class $c_1'(D)^\vee = \pi^*\bar{c}_1'(D) \in H^1(T^*Y, \mathcal{T}_{T^*Y})$. One has $H^j(T^*Y, \mathcal{O}_{T^*Y}) = \bigoplus\limits_i H^j(Y, S^i\mathcal{T}_Y)$, so we have $\delta : H^j(T^*Y, \mathcal{O}_{T^*Y}) \to H^{j+1}(T^*Y, \mathcal{O}_{T^*Y})$. Clearly δ coincides with the product with $\bar{c}_1'(D)^\vee$ via the map $\mathcal{T} \otimes_{\mathbb{C}} \mathcal{O} \to \mathcal{O}$, $\partial \times f \mapsto \partial(f)$. □

2.1.4 Example: Let Y be a compact complex torus, or an abelian variety. Then $\det \Omega^1_Y \simeq \mathcal{O}_Y$, hence $c_1'(D) = c_1(D)$. One has $H^1(Y, \Omega^1_Y) = H^0(Y, \Omega^1_Y) \otimes H^1(Y, \mathcal{O}) = \mathrm{Hom}(H^0(Y, \mathcal{T}_Y), H^1(Y, \mathcal{O}_Y))$ We will say that a class $\bar{c} \in H^1(Y, \Omega^1_Y)$ is non-degenerate if the map $\bar{c} : H^0(Y, \mathcal{T}_Y) \to H^1(Y, \mathcal{O}_Y)$ is isomorphism; a class $c \in F^1 H^1_{DR}(Y)$ is non-degenerate if such is an element $\bar{c} = c \bmod F^2 H^2_{DR}$ of $H^1(Y, \Omega^1_Y)$.

2.1.5 Lemma. *Y is D-rational iff $c_1(D)$ is non-degenerate.*

Proof: Note that $H^j(Y, S^i\mathcal{T}_Y) = S^i H^0(Y, \mathcal{T}_Y) \otimes \Lambda^j H^1(Y, \mathcal{O}_Y)$. This isomorphism identifies the complex from 2.1.2 with i-th symmetric power of the 2-term complex $H^0(Y, \mathcal{T}_Y) \xrightarrow{\bar{c}_1(D)} H^1(Y, \mathcal{O}_Y)$. Hence if $c_1(D)$ is non-degenerate then the conditions of 2.1.2 hold (the Koszul complex is acrylic, and Y is D-rational. If $c(D)$ is degenerate, the exact sequence $0 \to \mathbb{C} \to H^0(Y, \widetilde{\mathcal{T}}_D) \to H^0(Y, \mathcal{T}_Y) \xrightarrow{\bar{c}_1(D)} H^1(Y, \mathcal{O}_Y)$ shows that $\mathbb{C} \subsetneqq H^0(Y, \widetilde{\mathcal{T}}_D) \subset H^0(Y, D)$, so Y is not D-rational. □

2.1.6 Remark. Let $\mathcal{L}$ be a line bundle on a compact complex torus Y. If $c_1(\mathcal{L})$ is non-degenerate, then all the cohomologies $H^j(Y, \mathcal{L})$ vanish but a single one. We do not know whether one has a similar statement for a line bundle $\mathcal{L}$ on arbitrary $D_{\mathcal{L}}$-rational variety.

2.2 Canonical D-Connections

Assume we are in a situation 1.2, so we have a smooth morphism $p : Y \to S$ of smooth varieties, and D_Y is a tdo on Y.

2.2.1 Definition. *We will say that p is D_Y-rigid if one has $p_* D_{Y/S} = \mathcal{O}_S$ and for any $\tau \in \mathcal{T}_S$ there exists (locally along S) an element $\tilde{\tau} \in p_* D_Y$ such that for $f \in \mathcal{O}_S$ one has $p^* \tau(f) = [\tilde{\tau}, p^* f]$.* □

2.2.2 Lemma. *Consider the short exact sequence $0 \to \widetilde{\mathcal{T}}_{D_{Y/S}} \to \widetilde{\mathcal{T}}_{D_Y} \to p^* \mathcal{T}_S \to 0$. It defines the morphism $KS : \mathcal{T}_S \to R^1 p_* \widetilde{\mathcal{T}}_{D_{Y/S}}$ (the Kodaira-Spencer class). Then p is D-rigid iff $p_* D_{Y/S} = \mathcal{O}_S$ and the composition $\mathcal{T}_S \stackrel{KS}{\to} R^1 \pi_* \widetilde{\mathcal{T}}_{D_{Y/S}} \to R^1 p_* D_{Y/S}$ equal to 0.*

Proof. Clear. □

Assume that p is D-rigid. Let $\widetilde{\mathcal{T}}_S$ be the sheaf of all pairs $(\tau, \tilde{\tau})$ from 2.2.1. We have a short exact sequence $0 \to \mathcal{O}_S \stackrel{i}{\to} \widetilde{\mathcal{T}}_S \stackrel{\sigma}{\to} \mathcal{T}_S \to 0$, $i(f) = (o, f)$, $\sigma(\tau, \tilde{\tau} = \tau)$. Also $\widetilde{\mathcal{T}}_S$ carries an obvious Lie algebra and $\mathcal{O}_S$-module structure, so $\widetilde{\mathcal{T}}_S$ is an $\mathcal{O}$-extension of $\mathcal{T}_S$. Let $D_S = D_{\widetilde{\mathcal{T}}_S}$ be the corresponnding tdo (see A1.4). The map $\widetilde{\mathcal{T}}_S \to p_* D_Y$, $(\tau, \tilde{\tau}) \mapsto \tilde{\tau}$, extends (uniquely) to p-morphism $\alpha : D_S \to D_Y$. We will call α a canonical p-morphism, and the corresponding D_Y-connection ∇_{D_Y} a canonical D_Y-connection.

2.2.3 Lemma. (i) *A canonical D_Y-connection is actually a unique D_Y-connection for p.*

(ii) *A degree of ∇_α is equal to minimal degree of $\tilde{\tau}$ for $(\tau, \tilde{\tau}) \in \widetilde{\mathcal{T}}_S$ minus 1.*

(iii) *Let $L.$ be the filtration by degree along S on D_Y (see 1.2.5). One has* $\widetilde{\mathcal{T}}_S = p_* L_1$.

(iv) *Any (compatible) Lie algebra action on Y, S, D_Y preserves D_S and ∇_α (see 1.4.1).*

Proof: Clear. $\square$

2.2.4 Proposition. *Let $p : Y \to S$ be any smooth surjective morphism and D_Y be a tdo on Y such that for each $s \in S$ the fiber Y_s is D_{Y_s}-rational. Then p is D_Y-rigid, and one has $p_* D_Y = D_S$, $R^i p_* D_Y = 0$ for $i > 0$. If, moreover, D_Y satisfies conditions 2.1.2, then a canonical D_Y-connection has order 1.*

Proof. One has $\mathcal{O}_S \underset{\sim}{\to} p_* D_{Y/S}$, $R^i p_* D_{Y/S} = 0$ since $D_{Y/S}$ is flat $\mathcal{O}_S$-module and we have fiberwise rationality. Consider the filtration $L.$ on D_Y. Since $L_i / L_{i-1} = D_{Y/S'} \otimes S^i \mathcal{T}_S$ one has $R p_* L_i / L_{i-1} = S^i \mathcal{T}_S$. This implies that $R^i p_* D_Y = 0$ for $Ki > 0$ and $p_* D_Y$ is a tdo with a canonical filtration equal to $p_* L_i$. By 2.2.3 we see that p is D_Y-rigid and $p_* D_Y = D_S K$. $\square$

2.3 Canonical Quantization

Let $(X \xrightarrow{\pi} Y; \omega; \nabla_S)$ be an S-Lagrangian triple of order 1, D_Y be a corresponding tdo on Y.

2.3.1 Definition. *We will say that our Lagrangian triple is canonically quantizable if $p_Y : Y \to S$ is D_Y-rigid and a canonical D_Y-connection ∇_{D_Y} is a quantization (see 1.3).* $\square$

In this case ∇_{D_Y} (which is a unique D_Y-connection for p_Y) is called a canonical quantization of our triple. By 2.2.3(iv) ∇_{D_Y} is preserved by any symmetries of the triple. In some cases the compatibility 1.3 holds automatically, e.g., one has

2.3.2 Lemma. *Assume that for each $s \in S$ one has $H^0(Y_s, \mathcal{O}_{Y_s} = \mathbb{C}$, $H^0(Y_s, \mathcal{T}_{Y_s}) = 0$ and the maps $\delta_{D_{Y/S}} : H^0(Y_s, S^i\mathcal{T}_{Y_s}) \to H^1(Y_s, S^{i-1}\mathcal{T}_{Y_s})$ are injective for $i > 1$. Then our Lagrangian triple is canonically quantizable iff the composition $\mathcal{T}_S \overset{KS}{\to} R^1\pi_{Y_*}\widetilde{\mathcal{T}}_{D_{Y/S}} = R^1p_{Y_*}D_{D/S^1} \to R^1p_{Y_*}D_{Y/S^2}$ vanishes.*

Proof: Our conditions obviously imply that $p_{Y_*}D_{Y/S} = \mathcal{O}_S$, $p_{Y_*}(D_{Y/S}/\mathcal{O}_Y) = 0$. By 2.2.2, the above map $\mathcal{T}_S \to R^1p_{Y_*}D_{Y/S^2}$ vanishes iff p_Y is D_Y-rigid and a canonical D_Y-connection ∇_{D_Y} has order 1. By 1.3.2(ii) ∇_{D_Y} is a quantization. $\square$

2.3.3 Remark. Let $\pi : X \to Y$ be a morphism of S-varieties and $\omega \in \Omega^2_{X/S}$. Assume that these data satisfy conditions 1.1.3(i)–(ii). Note that the sheaf p_*-conn$^\omega$ of those p_X-connections ∇_S that ω is ∇_S-horizontal is an $\Omega^1_S \otimes p_{X_*}\mathcal{T}^\omega_{X/S}$-torsor (where $\mathcal{T}^\omega_{X/S} \subset \mathcal{T}_{X/S}$ is a subsheaf of vector fields that preserve ω). Therefore in case $p_{X_*}\mathcal{T}^\omega_{X/S} = 0$ there exists at most one such ∇_S which is automatically integrable (since the curvature lies in $\Omega^2_S \otimes p_{X_*}\mathcal{T}^\omega_{X/S} = 0$). Hence $(\pi : X \to Y, \omega, \nabla_S)$ is an S-Lagrangian triple. We will call such triples canonical Lagrangian triples.

2.4 Example: Heat Equation for θ-Functions

Let Y be a complex torus or an abelian variety. Denote by (-1) the involution $y \mapsto -y$ of Y. Let D_Y be a tdo on Y.

2.4.1 Definition. *A symmetric structure on* D_Y *is an isomorphism* $D_Y \xrightarrow{\alpha} (-1)^* D_Y$*. A symmetric tdo is a tdo equipped with a symmetric structure.*$\square$

A symmetric tdo forms a category $TDOS(Y)$ in an obvious manner. Certainly, we may repeat the above definition of symmetric structure for $\Omega_Y^{\geq 1}$-torsors or twisted cotangent bundles.

2.4.2 Lemma. *Any tdo admits a symmetric structure. A symmetric tdo* (D_Y, α) *has no automorphisms. One has* $(-1)^*(\alpha) \circ \alpha = id_{D_Y}$*, so* D_Y *is a* $\mathbb{Z}/2$*-equivariant tdo. Two symmetric tdo's are isomorphic iff they are isomorphic as usual tdo's.*

Proof: Follows from A1.6, A1.13 since (-1) acts on $H^1(Y, \Omega_Y^{\geq 1}) = F^1 H^2_{DR}) \subset H^2_{DR}(Y)$ as identity map, and on $H^0(Y, \Omega_Y^{\geq 1}) = F^1 H^1_{DR}(Y) \subset H^1_{DR}(Y)$ as minus identity. $\square$

2.4.3. We see that c_1 defines equivalence between $TDOS(Y)$ and a discrete category with the set of objects $F^1 H^2_{DR}(Y)$. For $c \in F^1 H^2_{DR}(Y)$ we will denote by D_c the corresponding symmetric tdo, and by $(\pi_c : X_c \to Y; \omega_c)$, the symmetric twisted cotangent bundle. Note that if c lies in $F^2 H^2_{DR}(Y) = H^0(Y, \Omega^2_Y)$ then the tdo D_c carries a unique symmetric (in an obvious sense) connection ∇_c with curvature c (cf. A1.7).

2.4.4. Here is an explicit construction of the twisted cotangent bundle X_c for a non-degenerate $c \in F^1 H^2_{DR}(Y)$. Let $0 \to H^1(Y, \mathcal{O}_Y)' \to X \xrightarrow{\pi} Y \to 0$ be a universal extension of Y (see, e.g. [MM]); we consider here the vector space $H^1(Y, \mathcal{O}_Y)'$ as an algebraic group). So X is a commutative algebraic group with Lie algebra Lie X canonically identified with $H^1_{DR}(Y)'$. One may describe points of X as line bundles with connection on a dual abelian variety Y^0; in the analytic case one identifies X with $H_1(Y, \mathbb{C}/\mathbb{Z})$. Our class $c \in F^1 H^2_{DR}(Y) \subset H^2_{DR}(Y) = \Lambda^2 H^1_{DR}(Y)$ defines an invariant 2-form ω_c on X; this form is closed, non-degenerate (since such was c), and π is a polarization for ω_c (since $c \in F^1 H^2_{DR}$), so $X_c = \{(\pi : X \to Y; \omega_c)\}$ is a twisted cotangent bundle on Y. The involution $(-1)_x : x \mapsto -x$ is a symmetric structure on X_c. Since $\pi^* : H^2_{DR}(Y) \underset{\sim}{\to} H^2_{DR}(X)$ is isomorphism, and $\pi^* X_c$ carries a section with curvature ω_c, we see that $c_1(X_c) = c$.

2.4.5. Now let $p_Y : Y \to S$ be an abelien scheme over S, i.e., a family Y_s, $s \in S$, of abelian varieties (so we are in an algebraic situation). Let c be a horizontal section of $\mathcal{H}^2_{DR}(Y/S)$ (with respect to Gauss-Manin connection) that lies in $F^1\mathcal{H}^2_{DR}(Y/S)$. For any $s \in S$ the element $c_s \in F^1 H^2_{DR}(Y_s)$ defines a symmetric twisted cotangent bundle $(\pi_{cs} : X_{cs} \to Y_s, \omega_{cs})$, in a canonical way. These spaces form a relative symmetric twisted cotangent bundle $\pi_c : X \to Y$, $\omega_c \in H^0(X, \Omega^{2c\ell}_{X/S})$, $(-1)_X : X_c \to X_c$. We will say that c is non-degenerate if for some (or any) $s \in S$ the class $c_s \in F^1\mathcal{H}^2_{DR}(Y_s)$ is non-degenerate.

2.4.6 Proposition. *Assume that c is non-degenerate. Then*

(i) $p_X = p_Y \circ \pi_c : X_c \to S$ *admits a unique symmetric connection* ∇_S *(i.e., the one such that* $(-1)_X \nabla_S = \nabla_S$*).*

(ii)$(\pi_c : X_c \to Y; \omega_c; \nabla_S)$ *is an S-Lagrangian triple which is canonically quantizable.*

Proof: (i) One has $H^0(X_s, \mathcal{O}_{X_s}) = \mathbb{C}, H^i(X_s, \mathcal{O}_{X_s}) = 0$ for $i > 0$ (to see this note that $H^{\cdot}(X_s, \mathcal{O}_{X_s}) = H^{\cdot}(H_s, \pi_{s*}\mathcal{O}_{X_s})$ since π_s is affine; the standard filtration A_i on $A = \pi_{S*}\mathcal{O}_X$ gives a spectral sequence with first term equal to Koszul complex, cf. 2.1.5). The connections for p_X form a $p_X^*\Omega_S^1 \otimes \mathcal{T}_{X/S}$-torsor on X. Since $\mathcal{T}_{X_s} = H^1_{DR}(Y_s) \otimes \mathcal{O}_{X_s}$ (see 2.4.3) we see that connections for p_X (global along the fibers of p_X) exist and form an $\mathcal{H}^1_{DR}(Y/S) \otimes \Omega^1_S$-torsor. Since $(-1)_X$ acts on $\mathcal{H}^1_{DR}(Y/S)$ as multiplication by -1, we see that there exists a unique symmetric connection ∇_S.

(ii) Note that X is naturally a group scheme over S. It follows easily by unicity that ∇_S is actually a unique connection for p_X compatible with group structure on X, and the induced connection on Lie $X/S = \mathcal{H}^1_{DR}(Y/S)'$ is (dual to) Gauss-Manin connection (see [MM]). Also ∇_S is flat. Since ω_c is invariant 2-form on X-horizontal with respect to Gauss-Manin connection (see 2.4.3) we see that it is ∇_S-horizontal, so $(\pi : X \to Y; \omega_c; \nabla_S)$ is an S-Lagrangian triple. By 2.1.5, 2.2.4, p_Y is D_Y-rigid and a canonical connection ∇_{D_Y} has order 1. Since ∇_{D_Y} is symmetric (being unique) the section $\nabla_S^0 - \tilde{\sigma}\nabla^0_{D_Y} \in \Omega^1_S \otimes p_{Y_*}(A_2/A_0) = \Omega^1_S \otimes p_{Y_*}(A_1/A)$ is symmetric (see 1.3.2), hence vanishes. By 1.3.2(ii) this means that ∇_{D_Y} is a quantization. $\square$

2.4.7 Remark: (i) If we are in an analytic situation, i.e., $p_Y : Y \to S$ is a family of compact complex tori, then 2.4.5 remains valid with the only correction: in (i) one should also demand that ∇_S has finite order (i.e., for $f \in \mathcal{O}_Y$, $t \in \mathcal{T}_S$ the function $\nabla_S(\tau)(f) \in \mathcal{O}_X$ should be polynomial along the fibers of π). The connection ∇_S defines an obvious "topological" local trivialization of the fibration $X = \mathcal{H}_1(Y/S, \mathbb{C}/\mathbb{Z}) \to Y$. $\square$

(ii) In the language of 2.3.4 the above Proposition 2.4.6(i) says that $(X/\pm 1 \xrightarrow{\pi} Y/\pm 1, \omega, \nabla_S)$ is a canonical S-Lagrangian triple. Here $/\pm 1$ means quotient modulo the involution (-1) which is an S-family of smooth "orbifolds" or "stack."

2.4.8. Assume that our class c is integral, i.e., $c_s \in H^2(Y_s, \mathbb{Z}(1))$. Localizing S, if necessary, one finds a symmetric line bundle $\mathcal{L}_c$ on Y together with a trivialization $e^*\mathcal{L}_c \simeq \mathcal{O}_S$ of its restriction to zero section e of Y. Put $\lambda = e^*\Omega = p_{Y_*}\Omega$, and choose a square-root of λ, i.e., a line bundle $\lambda^{1/2}$ on S together with isomorphism $\lambda^{1/2\otimes 2} = \lambda$. Then $\mathcal{L}_c \otimes p_Y^*\lambda^{1/2}$ is a Kostant line bundle, and a corresponding integrable projective connection on $R^i p_{Y_*}$ is a classicial heat equation for θ-functions.

5. Centralizers of Regular Elements

Let G be a connected reductive group, $\mathfrak{g}$ its Lie algebra, $\bar{G} := G/$ center G be the adjoint group, and $\mathcal{B}$ the variety of Borel subalgebras of $\mathfrak{g}$. We can also interpret $\mathcal{B}$ as the variety of Borel subgroups of G. Recall the definition of the Cartan group of G, the Cartan Lie algebra and the Weyl group. The action of $\bar{G}$ on $\mathcal{B}$ is transitive, so for each pair B_1, B_2 of Borel subgroups we may choose $g \in \bar{G}$ such that $\mathrm{Ad}(g)B_1 = B_2$ which induces the isomorphism $\mathrm{Ad}(g) : B_1/[B_2, B_1] \underset{\sim}{\to} B_2/[B_2, B_2]$. In fact, this isomorphism does not depend on a choice of g, hence we may identify canonically all the toruses $B/[B, B]$. This torus H is called the Cartan group of G. Its Lie algebra $\mathfrak{h}$ is called Cartan Lie algebra of $\mathfrak{g}$; one has a canonical isomorphism $\mathfrak{h} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ for $\mathfrak{b} \in B$. Put $\Gamma := \mathrm{Hom}(G_m, H)$. One defines similarly the Weyl group $W = W(G)$; it acts on H and $\mathfrak{h}$ in a canonical way. We also have the root data; denote by Δ the set of roots, and by $S \subset \Delta$ the subset of simple roots.

Denote by $p : \mathfrak{h} \to \mathfrak{h}/W := Y$ the projection, and by $R \subset Y$ the ramification locus of p. We have a canonical $\mathrm{Ad}G$-invariant projection $f : \mathfrak{g} \to Y$ such that $f/\mathfrak{b}$ coincides with the composition $\mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = \mathfrak{h} \xrightarrow{p} Y$ for any $\mathfrak{b} \in B$.

5.1. Let $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ be the open subset of regular (not necessarily semi-simple) elements of $\mathfrak{g}$. Put $\tilde{\mathfrak{g}}_{\mathrm{reg}} := \{(a, \mathfrak{b}) : a \in \mathfrak{b}\} \subset \mathfrak{g}_{\mathrm{reg}} \times \mathcal{B}$. Then $\tilde{\mathfrak{g}}_{\mathrm{reg}}$ is a smooth variety, and the projection $p' : \tilde{\mathfrak{g}}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}}$ is finite. The group $\bar{G}$ acts on these objects in an obvious manner. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}}_{\mathrm{reg}} & \stackrel{\tilde{f}_{\mathrm{reg}}}{\longrightarrow} & \mathfrak{h} \\ \downarrow p' & & \downarrow p \\ \mathfrak{g}_{\mathrm{reg}} & \stackrel{f_{\mathrm{reg}}}{\longrightarrow} & Y \end{array}$$

where $f_{\mathrm{reg}} := f|_{\mathfrak{g}_{\mathrm{reg}}}$, and $f_{\mathrm{reg}}(a, \mathfrak{b}) = a \mod [\mathfrak{b}, \mathfrak{b}] = \mathfrak{h}$. One knows (see [K2]) that

(i) this diagram is Cartesian, hence W acts along the fibers of p', and $\mathfrak{g}_{\mathrm{reg}} = W\backslash\tilde{\mathfrak{g}}_{\mathrm{reg}}$.

(ii) f_{reg} is a smooth projection. The adjoint action of $\bar{G}$ is transitive along the fibers of f_{reg}. Hence $Y = \bar{G}\backslash\mathfrak{g}_{\mathrm{reg}}$, $\mathfrak{h} = \widetilde{Y} =: \bar{G}\backslash\tilde{\mathfrak{g}}_{\mathrm{reg}}$.

(iii) f_{reg} admits a global section $s : Y \to \mathfrak{g}_{\mathrm{reg}}$.

5.2 Let $a \in \mathfrak{g}_{\mathrm{reg}}$ be a regular element. Denote by $\mathcal{H}_a$ the centralizer of a, and by $i_a : \mathcal{H}_a \hookrightarrow G$ the embedding. One knows that $\mathcal{H}_a$ is a commutative group of dimension $\dim \mathfrak{h}$. For any Borel subgroup $B \subset G$ such that $a \in \mathfrak{b} = \mathrm{Lie}\, B$ one has $\mathcal{H}_a \subset B$, hence the projection $B \to B/[B, B] = H$ defines the morphism $\varphi_B : \mathcal{H}_a \to H$. For $\tilde{a} = (a, \mathfrak{b}) \in \tilde{\mathfrak{g}}_{\mathrm{reg}}$ we put $\varphi_{\tilde{a}} := \varphi_B : \mathcal{H}_a \to H$. If a is a regular semi-simple, then $\varphi_{\tilde{a}}$ is an isomorphism.

One may describe $\mathcal{H}_a$ as follows. Let $M_a \subset G$ be the centralizer of a_{ss} (:= semi-simple part of a). This is a Levi subgroup of G. Since $a \in \mathrm{Lie}\, M_a$, one has Center $M_a \subset \mathcal{H}_a$. In fact, Center M_a coincides with the reductive part of $\mathcal{H}_a$: if $\mathcal{H}_{aun}$ denotes the unipoint radical of $\mathcal{H}_a$ then one has $\mathcal{H}_a =$ Center $M_a \times \mathcal{H}_{aun}$. Note that $\mathcal{H}_{aun} = \ker \varphi_{\tilde{a}}$, hence Center $M_a \underset{\sim}{\to} \varphi_{\tilde{a}}(\mathcal{H}_a)$.

We will need a bit of information on the structure of the group $\mathcal{H}_a/\mathcal{H}_a^0 =$ Center $M_a/\mathrm{Center}^0 M_a$ of connected components of $\mathcal{H}_a$. For a root γ let $\gamma^\vee \in \Gamma$

be the corresponding co-root, and $\sigma_\gamma \in W$ be the reflection $a \mapsto a - \gamma(a)\gamma^\vee$; let $\chi_\gamma : H \to G_m$, $i_\gamma : G_m \to H$ be the corresponding character and 1-parameter subgroup. We will say that γ is a type 1 root if $\Gamma = \Gamma^{\sigma_\gamma} \oplus \mathbb{Z}\gamma^\vee$, that γ is of type 2 if $\Gamma = \Gamma^{\sigma_\gamma} \oplus \frac{1}{2}\mathbb{Z}\gamma^\vee$, and that γ is of type 3 in other cases (in other words, γ is of type 3 if the projection $\Gamma^{\sigma_\gamma} \to \Gamma_{\sigma_\gamma}$ from σ_γ-invariants to σ_γ-coinvariants is isomorphism). Therefore γ is of type 2 if the rank 1 subgroup that corresponds to γ equal to PGL_2. Denote by $S_i \subset S$, $i = 1, 2, 3$, the subset of simple roots of type i.

If M_a is our Levi subgroup and B is a Borel subgroup such that $a_{ss} \in \mathfrak{b} = \mathrm{Lie}\ B$, then $B_a := B \cap M_a$ is a Borel subgroup of M_a and $B_a/[B_a, B_a] = B/[B, B]$. Hence a choice of B defines the isomorphism between the Cartan groups of M_a and G, and identifies the root system of M_a with the subsystem of the one of G. In particular, S_a (:= simple roots of M_a) $\subset S$, and $W_{S_a} \subset W$ is the Weyl group of M_a.

5.2.1 Lemma. (i) *One has Center* $M_a = \bigcap_{\gamma \in S_a} \ker \chi_\gamma$, $H^{W_{S_a}} = \bigcap_{\gamma \in S_a} \ker(i_\gamma \chi_\gamma)$,

(ii) *One has* $H^{W_{S_a}}/$*Center* $M_a \xrightarrow{\sim} \mathbb{Z}/2^{S_{a2}}$, *where* $A_{a2} := S_a \cap S_2$.

(iii) *In each orbit of* S_{S_a} *in the roots of* M_a *there is at most one simple root.*

(iv) *If* $a \in \mathrm{Lie}\ B = \mathfrak{b}$, *then* W_{S_a} *equals the stabilizer of* $\tilde{a} = (a, \mathfrak{b}) \in \tilde{\mathfrak{g}}_{reg}$ *with respect to* W*-action (see 5.1(i)).*

(v) *If* $S_a = \{\gamma\}$, *then the group* $\mathcal{H}_q/\mathcal{H}_q^0$ *equals* $\mathbb{Z}/2$ *if* γ *is of type 1 and is trivial otherwise.*

Proof: Easy, e.g., morphism $(\chi_\gamma) : H \to \prod_{\gamma \in S_a} G_m$ is surjective and $\ker i_\gamma$ is $\{\pm 1\}$ if γ is of type 2 and trivial otherwise. Therefore the map $(\chi_\gamma)_{\gamma \in S_{a2}}$

defines the isomorphism $\bigcap_{\gamma\in S_a}\ker(i_\gamma\chi_\gamma)/\bigcap_{\gamma\in S_a}\ker(\chi_\gamma) \underset{\sim}{\to} (\pm 1)^{S_{a2}}$, hence (ii) follows from (i). The morphism $\chi_\gamma : H^W \to \{\pm 1\}$ depends only on the W-orbit of γ, hence (iii) follows from (ii). $\square$

5.3. When $a \in \mathfrak{g}_{\mathrm{reg}}$ varies the groups $\mathcal{H}_a$ form a flat commutative group scheme $\mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}}$ on $\mathfrak{g}_{\mathrm{reg}}$ equipped with the embedding $i : \mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}} \hookrightarrow G_{\mathfrak{g}_{\mathrm{reg}}}$ to the constant group scheme G on $\mathfrak{g}_{\mathrm{reg}}$. The morphisms $\varphi_{\tilde{a}}$ form a canonical morphism $\varphi_{\tilde{\mathfrak{g}}_{\mathrm{reg}}} : \mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}} := p^{'*}\mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}} \to \mathcal{H}_{\tilde{\mathfrak{g}}_{\mathrm{reg}}}$. The W-action on $\tilde{\mathfrak{g}}_{\mathrm{reg}}$ lifts to our group schemes: namely, W acts on $\mathcal{H}_{\tilde{\mathfrak{g}}_{\mathrm{reg}}}$ in an obvious manner, and on $\mathcal{H}_{\tilde{\mathfrak{g}}_{\mathrm{reg}}} = H \times \tilde{\mathfrak{g}}_{\mathrm{reg}}$ in a diagonal one. The morphism φ commutes with W-action.

All the picture is equivariant with respect to (adjoint) action of G on all our schemes. Note that the stabilizer of a point $a \in \mathfrak{g}_{\mathrm{reg}}$, equal to the image of $\mathcal{H}_a$ in $\bar{G}$, acts on the fiber $\mathcal{H}_a$ trivially (since $\mathcal{H}_a$ is commutative). Therefore, according to 5.1(ii), the scheme $\mathcal{H}_{\tilde{\mathfrak{g}}}$ descents to Y: we have a canonical group scheme $\mathcal{H}_Y$ on Y such that $\mathcal{H}_{\tilde{\mathfrak{g}}_{\mathrm{reg}}} = f^*\mathcal{H}_Y$. For any section s of f_{reg} one has a canonical isomorphism $s^*\mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}} = \mathcal{H}_Y$, hence the embedding $i_s := s^*(i) : \mathcal{H}_Y = s^*\mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}} \to s^*G_{\mathfrak{g}_{\mathrm{reg}}} = G_Y$.

The morphism $\varphi_{\mathfrak{g}_{\mathrm{reg}}}$ descents to a canonical morphism $\varphi_{\widetilde{Y}} : \mathcal{H}_{\widetilde{Y}} := p^*\mathcal{H}_Y \to H_{\widetilde{Y}}$ equivariant with respect to W-action. By adjointness we have the morphism $\varphi_Y : \mathcal{H}_Y \to (p_* H_{\widetilde{Y}})^W$. This is an embedding which is isomorphism off R. As follows from 5.2.1, the cokernel of φ_Y is a constructible sheaf with a stalk at $y \in R$ equal to $\mathbb{Z}/2^{S_{y^2}}$, where $S_{y^2} \subset S_2$ is the set of type 2 simple roots "vanishing at y." In particular, $\Gamma(Y, \mathrm{Coker}\ \varphi_Y) = (\mathrm{Coker}\ \varphi_Y)_0 = \mathbb{Z}/2^{S_2}$. Clearly $H^1(Y, \mathrm{Coker}\ \varphi_Y) = 0$.

Note that Center $G \subset \mathcal{H}_a$ for any $a \in \mathfrak{g}_{\mathrm{reg}}$, hence Center $G \subset \Gamma(Y, \mathcal{H}_Y)$.

Precisely, one has

5.3.1 Lemma. $\Gamma(Y,\mathcal{H}_Y) = \text{Center } G,\ H^1(Y,\mathcal{H}_Y) = 0$.

Proof: Note that all the global (algebraic) H-valued functions on $\widetilde{Y} = \mathfrak{h}$ are constant. Hence $\Gamma(Y,\mathcal{H}_Y) = \ker(\Gamma(Y,(p_*H_{\widetilde{Y}})^W) \to \mathbb{Z}/2^{S_2}) = \ker(H^W \to \mathbb{Z}/2^{S_2}) = \text{Center } G$ by 6.3. Now let $\mathcal{F}$ be any $\mathcal{H}_Y$-torsor, and $\widetilde{\mathcal{F}} := \varphi_{\widetilde{Y}}(p^*\mathcal{F})$ be the corresponding W-equivariant H-torsor on $\widetilde{Y}$. Since any H-torsor on $\widetilde{Y}$ is trivial, the value at 0 map defines the isomorphism $\Gamma(\widetilde{Y},\widetilde{\mathcal{F}}) \underset{\sim}{\to} \widetilde{\mathcal{F}}_{(0)}$. Therefore for the $(p_*H_{\widetilde{Y}})^W$-torsor $\varphi_Y(\mathcal{F}) = (p_*\widetilde{\mathcal{F}})^W \supset \mathcal{F}$ one has $\Gamma(Y,\varphi_{Y^\cdot}(\mathcal{F})) = \Gamma(\widetilde{Y},\widetilde{\mathcal{F}})^W = \widetilde{\mathcal{F}}^W_{(0)}$, and $\Gamma(Y,\mathcal{F}) = \mathrm{Im}(\varphi_{\widetilde{Y}_0} : \mathcal{F}_0 \to \widetilde{\mathcal{F}}^W_{(0)} \neq \emptyset$, q.e.d. $\square$

5.4. Consider the canonical embedding $i : \mathcal{H}_{\mathfrak{g}_{\text{reg}}} \hookrightarrow G_{\mathfrak{g}_{\text{reg}}}$. We would like to descent it down to Y. We assume that $\bar{G}$ acts on $G_{\mathfrak{g}_{\text{reg}}} = G \times \mathfrak{g}_{\text{reg}}$ by a diagonal adjoint action. Then i is $\bar{G}$-equivariant. Note that the stabilizer of a point $a \in \mathfrak{g}_{\text{reg}}$ acts on a fiber G_a in a nontrivial way; hence we need for $G_{\mathfrak{g}_{\text{reg}}}$ a bit more clever descent then the obvious one used for $\mathcal{H}_{\mathfrak{g}_{\text{reg}}}$ in 5.3.

Namely, Π denotes the set of global sections $s : Y \to \mathfrak{g}_{\text{reg}}$ of $f_{\mathfrak{g}_{\text{reg}}}$; according to 5.1(iii) Π is a nonempty $\bar{G}(Y)$-set.

5.4.1 Lemma. Π *is a* $\bar{G}(Y)$*-torsor.*

Proof: For $s_1, s_2 \in \Pi$ consider the sheaf $\phi_{s_2 s_1}$ on Y, defined by formula $\phi_{s_2 s_1}(U) := \{g \in G(U) : \mathrm{Ad}(g)s_{1|U} = s_{2|U}\}$. This is an $\mathcal{H}_Y$-torsor with respect to right multiplication by $i_{s_1} : \mathcal{H}_Y \hookrightarrow G_Y$. By 5.3.1 the global sections $\Gamma(Y,\phi_{s_2 s_1})$ form a torsor with respect to the action of Center $G = \Gamma(y,\mathcal{H}_Y)$.

Hence for any $s_1, s_2 \in \Pi$ there exists a unique element $g_{s_2 s_1} \in \bar{G}(Y) = G(Y)/\text{Center } G$ such that $\mathrm{Ad}(g)s_1 = s_2$.We are done. $\square$

Denote by $G_Y^\vee$ the group scheme on Y obtained from G_Y by Π-twist (with respect to adjoint action of $\bar{G}(Y)$). Hence for any $s \in \Pi$ we have a canonical isomorphism $j_s : G_Y^\vee \underset{\sim}{\to} G_Y$ such that $k_{s_2} j_{s_1}^{-1} = \mathrm{Ad}(g_{s_2 s_1})$. There is a canonical embedding $i : \mathcal{H}_Y \hookrightarrow G_Y^\vee$ such that $j_s i = i_s$. Note that we have *no* canonical isomorphism between $f_{\mathrm{reg}}^* G_Y^\vee$ and $G_{\mathfrak{g}_{\mathrm{reg}}}$.

5.5. The variation considered in 5.1 also carry a natural Gm-action that commutes with $\bar{G}$- and W-actions. Namely, Gm acts on $\mathfrak{G}_{\mathrm{reg}}$ and $\mathfrak{h}$ by homotheties, and this determines the Gm-actions on $Y = W\backslash\mathfrak{h} - \bar{G}\backslash\mathfrak{g}_{\mathrm{reg}}$ and $\tilde{\mathfrak{g}}_{\mathrm{reg}} = \mathfrak{g}_{\mathrm{reg}} \times_Y \mathfrak{h}$. Explicitly, if p_i are homogeneous generators of $S(\mathfrak{h}^*)^W$ of degree d_i, so $(p_i) : Y \underset{\sim}{\to} \mathbb{C}^{\dim \mathfrak{h}}$, then Gm acts on Y in coordinates p_i by formula $\lambda(p_i) = (\lambda^{d_i} p_i)$.

This Gm-action lifts to our group schemes $\mathcal{H}_Y$, $G_Y^\vee$ and $H_{\widetilde{Y}}$. Namely, the Gm-action on $H_{\widetilde{Y}} = H \times \widetilde{Y}$ is the trivial one. For $a \in \mathfrak{g}_{\mathrm{reg}}$ and $\lambda \in \mathbb{C}^*$ one has $\mathcal{H}_a = \mathcal{H}_{\lambda a}$, which defines the Gm-action on $\mathcal{H}_{\mathfrak{g}_{\mathrm{reg}}}$ which descents down to $\mathcal{H}_Y$. The group Gm acts on the set Π of global sections of f_{reg} by formula $(\lambda s)(y) = \lambda s(\lambda^{-1} y)$; for $g = g(y) \in \bar{G}(Y)$ we have $\lambda(gs) = (\lambda g)(\lambda s)$, where $(\lambda g)(y) = g(\lambda^{-1} y)$. This defines the Gm-action on $G_Y^\vee$ such that $j_s \circ \lambda = j_{\lambda s} : G_Y^\vee \underset{\sim}{\to} G_Y$ for $s \in \Pi$, $\lambda \in \mathbb{C}^*$.

The morphisms $\varphi_{\widetilde{Y}} : \mathcal{H}_{\widetilde{Y}} \hookrightarrow G_{\widetilde{Y}}$, $i : \mathcal{H}_Y \hookrightarrow \mathcal{H}_Y^\vee$ commute with Gm-action. We will need to know whether there exists a Gm-equivariant G-torsor $\mathcal{T}$ such that the group $G_Y^\vee$ with Gm-action is isomorphic to Aut $\mathcal{T}$. Or, eqeuiva-lently, whether the Gm-equivariant $\bar{G}(Y)$-torsor Π lifts to a Gm-equivariant

$G(Y)$-torsor. Since $G(Y)$ is a central extension of $\bar{G}(Y)$ by Center G, the obstruction α for lifting an element $c\ell\,\Pi \in H^1(Gm, \bar{G}(Y))$ to $H^1(Gm, G(Y))$ lies in a finite group $H^2(Gm, \text{Center } G) = H^2(Gm, A) = A(-1)$, where A denotes the group of connected components of Center G (and (-1) is Tate twist).

The obstruction α could be easily computed. Namely, let α be a regular nilpotent element, and $\tilde{\nu} : SL_2 \to G$ be a morphism such that Lie $\tilde{\nu} = ((\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})) = a$ (so $\tilde{\nu}$ is a Kostant principal TDS). Let $\nu : Gm \to G$, $\nu(\lambda) = \tilde{\nu}\left((\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix})\right)$ be the corresponding one-parameter subgroup, so one has Ad $\nu(\lambda)(a) = \lambda^2 a$ and $\nu(-1) \in \text{Center } G$.

5.5.1 Lemma. *The obstruction α equals the image of $\nu(-1)$ in A. In particular $2\alpha = 0$.*

Proof: Put $\psi := \{\lambda a, \lambda \neq 0\} \subset \mathfrak{g}_{\text{reg}}$, $\Pi_a := \{s : s(0) \in \psi\} \subset \Pi$, $G_a := \nu(Gm) \cdot \text{Center } G$, $G(Y)_a := \{g \in G(Y) : g(0) \in G_a\} \subset G(Y)$, $\bar{G}_a = G_a/\text{Center } G \subset \bar{G}$, $\bar{G}(Y)_a := \{\bar{g} \in \bar{G}(Y) : \bar{g}(0) \in \bar{G}_a\} \subset \bar{G}(Y)$. Clearly, $G(Y)_a$, G_a are central extensions of the corresponding $^-$-groups by Center G. We have canonical morphisms $G(Y) \stackrel{\mu}{\hookleftarrow} G(Y)_a \stackrel{\pi}{\to} G_a$, $\pi(g) = g(0)$, identical on Center G, and the corresponding morphisms of $^-$-groups. Now Π_a, ψ are $\bar{G}(Y)_a$- and $\bar{G}$-torsors, respectively, and the obvious maps $\Pi \stackrel{\mu'}{\hookleftarrow} \Pi_a \stackrel{\pi'}{\to} \psi$, $\pi'(s) = s(0)$, are μ- and π-compatible. Note that all our groups and torsors carry an obvious Gm-action. Hence, by functoriality, the obstructions for lifting Π, Π_a and ψ to, respectively Gm-equivariant $G(Y)$-, $G(Y)_a$- and G_a-torsors. The obstruction for ψ coincides with the image of $\nu(-1)$ in A, and we are done. $\square$

Example: In case $G = SL_n$ the obstruction α vanishes iff n is odd.

5.6. Consider a pair $(\mathcal{T}, \sigma)$ where $\mathcal{T}$ is a Gm-equivariant G-torsor on Y, and $\sigma : \mathcal{T} \to \mathfrak{g}_{\mathrm{reg}}$ is a morphism of Y-schemes that commutes with $Gm \times G$-action. Such $(\mathcal{T}, \sigma)$'s form a groupoid $\hat{\Pi}$ (the morphisms between $(\mathcal{T}, \sigma)$'s are morphisms of Gm-equivariant G-torsors that commute with σ's).

5.6.1. For $(\mathcal{T}, \sigma) \in \hat{\Pi}$ the set $\hat{\Pi}_{\mathcal{T}} := \Gamma(Y, \mathcal{T})$ is nonempty (since the obstruction for lifting a section of $\mathfrak{g}_{\mathrm{reg}}$ to a one of $\mathcal{T}$ lies in $H^1(Y, \mathcal{H}_Y) = 0$), hence it is a Gm-equivariant $G(Y)$-torsor. Therefore $\sigma : \tilde{\Pi}_{\mathcal{T}} \to \Pi$ is a lifting of a Gm-equivariant $\bar{G}(Y)$-torsor Π to a Gm-equivariant $G(Y)$-torsor. Such liftings form a groupoid $\hat{\Pi}'$ in an obvious manner. Clearly, the above functor $\hat{\Pi} \to \hat{\Pi}'$, $(\mathcal{T}, \sigma) \to (\tilde{\Pi}_{\mathcal{T}}, \sigma)$, is equivalence of categories (the inverse functor assigns to $(\tilde{\Pi}, \sigma)$ the induced G_Y-torsor $\mathcal{T}_{\tilde{\Pi}} := G_Y \times_{G(Y)} \tilde{\Pi}$).

We see that $\hat{\Pi}$ is nonempty iff the obstruction α from 5.5.1 vanishes; assume for a while that this is the case.

5.6.2. Let $\mathcal{P}$ denote the category of Gm-equivariant Center G-torsors. This is a strictly commutative Picard category with automorphism group of an object equal to Center G, and the group of isomorphism classes of objects equal to $H^1(Gm, \text{Center } G) = \Gamma^W$. We have an obvious "multiplication of torsors" functor $* : \mathcal{P} \times \hat{\Pi} \to \hat{\Pi}$. It is clear (look at $\hat{\Pi}'$-version) that $*$ makes $\hat{\Pi}$ a "$\mathcal{P}$-torsor": for any $(\mathcal{T}, \sigma) \in \hat{\Pi}$ the corresponding functor $\mathcal{P} \to \hat{\Pi}$, $\mathcal{P} \mapsto P * (\mathcal{T}, \sigma)$, is the equivalence of categories. Equivalently, Π is a $\mathcal{P}$-gerb. The following lemma follows from the definitions:

5.6.3 Lemma. *For $(\mathcal{T}, \sigma) \in \hat{\Pi}$ we have a canonical Gm-equivariant iso-*

morphism $G_Y^\vee = \mathrm{Aut}\,\mathcal{T}$ *(:= automorphisms of* $\mathcal{T}$ *as* G_Y*-torsor). It identifies* $\mathcal{H}_Y \subset G_Y^\vee$ *with the subgroup* $\{\varphi\mathbb{R}, am\mathrm{Aut}\,\mathcal{T} : \sigma\varphi = \varphi\}$.

5.6.4. If the obstruction α from 5.5.1 does not vanish, let us consider the "squared" action of Gm on our spaces (the new action of $\lambda \in Gm$ is the old one of λ^2). We may repeat the above constructions for this action. The corresponding category $\hat{\Pi}^{(2)}$ of pairs $(\mathcal{T}^{(2)}, \sigma)$, where $\mathcal{T}^{(2)}$ is a Gm-equivariant (for a new action!) G-torsor on Y, and $\sigma : \mathcal{T}^{(2)} \to \mathfrak{g}_{\mathrm{reg}}$ is a $Gm \times G$-map, is nonempty by 5.5.1. We may repeat 5.6.1–5.6.3 word-by-word.

5.7. For a quantum analog of the above constructions, see [KL].

6. A Construction of G-Bundles

Let C be a smooth projective curve, and $\mathcal{L}$ be a line bundle on C. Denote by $\mathcal{L}^{\cdot} := \mathcal{L}\backslash\{\text{zero section}\}$ the corresponding Gm-torsor. If X is any variety with a Gm-action, then $X_{\mathcal{L}}$ denotes X twisted by $\mathcal{L}^{\cdot}$. Therefore $X_{\mathcal{L}}$ is a C-scheme equal to the quotient of $\mathcal{L}^{\cdot} \times X$ modulo Gm-action $\lambda(\ell, x) = (\lambda\ell, \lambda^{-1}x)$. In particular, if $X = V$ is a vector space with Gm-action by homotheties, then $V_{\mathcal{L}} = \mathcal{L} \otimes V$. An $\mathcal{L}$-twisted map $\theta : C \underset{\mathcal{L}}{\to} X$ is, by definition, a section of $X_{\mathcal{L}}$. Equivalently, this is a Gm-equivariant map $\theta_{\mathcal{H}} : \mathcal{L}^{\cdot -1} \to X$.

6.1. From now on assume that $\mathcal{L}$ is positive. Let $\theta : C \underset{\mathcal{L}}{\to} Y$ be a $\mathcal{L}$-twisted map (here Y carries the Gm-action defined in 5.5). We will say that θ is regular, if for any $c \in C$ such that $\theta(c) \in R$ one has $\theta_*(T_C(c)) \subset T_R(\theta(c))$. Equivalently, this means that the image of θ intersects R transversally at regular points of R.

Assume that θ is regular. Put $\tilde{C}_\theta := C \times_{Y_{\mathcal{L}}} \widetilde{Y}_{\mathcal{L}}$. This is a C-scheme with respect to projection $p : \tilde{C}_\theta \to C$ equipped with a W-action along the fibers of p. The projection $\tilde{C}_\theta \to \widetilde{Y}_{\mathcal{L}}$ is a W-equivariant $\mathcal{L}_{\tilde{C}_\theta}$-twisted map $\tilde{\theta} : \tilde{C}_\theta \underset{\mathcal{L}_{\tilde{C}_\theta}}{\longrightarrow} \widetilde{Y} = \mathfrak{h}$ which is the same as W-invariant section $\tilde{\theta}$ of $\mathcal{L}_{\tilde{C}_\theta} \otimes \mathfrak{h}$.

Lemma. (i) *$\tilde{C}_\theta$ is a smooth irreducible projective curve.*

(ii) *The W-action on $\tilde{C}_\theta$ is free at generic point of $\tilde{C}_\theta$, and $C = W\backslash\tilde{C}_\theta$.*

(iii) *The non-trivial stabilizers of points of $\tilde{C}_\theta$ are precisely all the order two subgroups $W_\gamma := \{1, \sigma_\gamma\} \subset W$, γ is a root.*

Proof: Let us prove that for any root γ one has $\tilde{C}_\theta^{\sigma_\gamma} \neq \emptyset$. Let $\mathfrak{h}^{\sigma_\gamma} \subset \mathfrak{h}$ be the corresponding hyperplane. Since $\mathcal{L}$ is positive, a section $\tilde{\theta}$ mod $\mathfrak{h}^{\sigma_\gamma}$ of $\mathcal{L}_{\tilde{C}_\theta} \otimes \mathfrak{h}/\mathfrak{h}^{\sigma_\gamma}$ must have a zero $x \in \tilde{C}_\theta$. Clearly, $x \in \tilde{C}_\theta^{\sigma_\gamma}$.

Let us prove that $\tilde{C}_\theta$ is connected. Let $\tilde{C}'_\theta$ be a connected component of $\tilde{C}_\theta$. The same reason as above shows that for any root γ one has $\tilde{C}'^{\sigma_\gamma}_\theta \neq \emptyset$, hence $\sigma_\gamma \tilde{C}'_\theta = \tilde{C}_\theta$. So $W\tilde{C}'_\theta = \tilde{C}'_\theta$. Since $W\tilde{C}'_\theta$ obviously equals $\tilde{C}_\theta$, we are done.

The other statements of the lemma are obvious. $\square$

6.2. Consider the pull-back of the group schemes $\mathcal{H}_Y$, $G_Y^\vee$ by the projection $\mathcal{L}' \times Y \to Y$. According to 5.5 they carry a canonical Gm-action, hence by descent we get the group scheme $\mathcal{H}_{Y_\mathcal{L}}$, $G^\vee_{Y_\mathcal{L}}$ on $Y_\mathcal{L}$ together with a canonical embedding $i : \mathcal{H}_{Y_\mathcal{L}} \hookrightarrow G^\vee_{Y_\mathcal{L}}$, $\varphi_{Y_\mathcal{L}} : p^*\mathcal{H}_{Y_\mathcal{L}} \to H_{\widetilde{Y}_\mathcal{L}}, \varphi_{Y_\mathcal{L}} : \mathcal{H}_{Y_\mathcal{L}} \to (p_* G_{\widetilde{Y}_\mathcal{L}})^W$.

6.2.1. Remark. $G^\vee_{Y_\mathcal{L}}$ is a twisted form of a constant group scheme $G_{Y_\mathcal{L}}$. If the obstruction α from 5.5.1 vanishes, then a choice of $(\tau, \sigma) \in \hat{\Pi}$ (see 5.6) defines, by Gm-descent, a $G_{Y_\mathcal{L}}$-torsor $\mathcal{T}_{Y_\mathcal{L}}$ with $G^\vee_{Y_\mathcal{L}} = \mathrm{Aut}\ \mathcal{T}_{Y_\mathcal{L}}$. If α is arbitrary, let us assume that deg $\mathcal{L}$ is even. Choose $\mathcal{L}^{1/2}$ (:= a Gm-torsor s.t. $(\mathcal{L}^{1/2})^2 = \mathcal{L}$). Then $Y_\mathcal{L} = Y^{(32)}_{\mathcal{L}^{1/2}}$, where $Y^{(2)}$ is Y with "squared" Gm-action. Now a choice of $\mathcal{T}^{(2)}$ in $\hat{\Pi}^{(2)}$ (see 5.6.4) defines, by Gm-descent from $\mathcal{L}^{1/2} \times Y^{(2)}$, a $G_{Y_\mathcal{L}}$-torsor $\mathcal{T}^{(2)}_{Y_\mathcal{L}}$ with $G^\vee_{Y_\mathcal{L}} = \mathrm{Aut}\ \mathcal{T}^{(2)}_{Y_\mathcal{L}}$.

Let $\theta : C \underset{\mathcal{L}}{\to} Y$ be a regular $\mathcal{L}$-twisted map. Put $\mathcal{H}_\theta := \theta^* \mathcal{H}_{Y_\mathcal{L}}$, $G^\vee_\theta := \theta^* G^\vee_{Y_\mathcal{L}}$; one has a canonical embedding $(p_* H_{\widetilde{C}_\theta})^W \overset{\varphi}{\leftarrow} \mathcal{H}_\theta \overset{i}{\to} G'_\theta$. The group scheme G'_θ is a twisted form of G_C; if deg $\mathcal{L}$ is even, or α vanishes, then following 6.2.1, we get a G_C-torsor $\mathcal{T}^{(2)}_\theta := \theta^* \mathcal{T}^{(2)}_{Y_\mathcal{L}}$, or $\mathcal{T}_\theta = \theta^* \mathcal{T}_{Y_\mathcal{L}}$ with $G^\vee_\theta$ identified with its automorphism sheaf.

Denote by $\mathcal{R}_\theta \subset C$ the ramification set for $p : \tilde{C}_\theta \to C$. To each point $x \in \mathcal{R}_\theta$ there corresponds a conjugacy class of roots γ_x, so that W_{γ_x} are stabilizers of points in $p^{-1}(x)$. We will say that $x \in \mathcal{R}_\theta$ is a type i ($i = 1, 2, 3$) point if γ_x is a type i root (see 5.2.1); let $\mathcal{R}_{\theta_i} \subset \mathcal{R}_\theta$ be a subset of type i points.

6.2.2 Lemma. (i) $(p_* H_{\tilde{C}_\theta})^W / \varphi(\mathcal{H}_\theta)$ *is a skyscraper sheaf* $\bigotimes_{x \in \mathcal{R}_{\theta_2}} \mathbb{Z}/2x$.

(ii) *The embedding i identifies global sections $\Gamma(C, \mathcal{H}_\theta)$ with Center G.*

Proof: (i) follows from 5.3. One has $\Gamma(C, (p_* H_{\tilde{C}_\theta})^W) = H(\tilde{C}_\theta)^W = H^W$ (since $\tilde{C}_\theta$ is connected and proper), hence (ii) follows from (i) and 5.2.1. $\square$

6.3. We are going to relate G-bundles on C and W-equivariant H-bundles on $\tilde{C}_\theta$ using $\mathcal{H}_\theta$-torsors as mediators.

6.3.1 Remark. Let G'_C be any twisted form of G_C. Then the categories of G'_C-torsors in Zariski, étale and classical topology on C are canonically equivalent. For Zariski = étale see [ ] (for $G \neq GL_n$ one really needs here that C is a curve), and étale = classical is GAGA-type statement. Similarly, $\mathcal{H}_\theta$-torsors are the same in Zariski, étale and classical versions.

Let $\Gamma_{\mathrm{root}} \subset \Gamma$ be the sublattice generated by coroots; note that W acts trivially on $\Gamma/\Gamma_{\mathrm{root}}$. Consider the i-induction functor between the stacks of torsors $i_{\mathrm{tors}} : \mathcal{H}_\theta - \mathrm{tors} \to G^\vee_\theta - \mathrm{tors}$.

6.3.2. Lemma. *The functor i_{tors} induces the bijection between the sets of connected components of stacks $\mathcal{H}_\theta$-tors and $G^\vee_\theta$-tors. These sets are in a natural 1-1 correspondence with $\Gamma/\Gamma_{\mathrm{root}}$.*

Proof: In the proof we willuse the analytic version of torsors.

1. Note that $\Gamma(1)$ coincides with the fundamental group $\pi_1(H)$. An embedding of a maximal torus $H \hookrightarrow G$ induces a canonical isomorphism $\Gamma/\Gamma_{\mathrm{root}}(1) \underset{\mathcal{L}}{\to} \pi_1(G)$. Consider the universal covering $\widetilde{G}$ of the topological groups G_{top}, therefore $\widetilde{G}$ is a central extension of G_{top} by $\pi_1(G)$. The adjoint action of $\bar{G}$ lifts to $\widetilde{G}$, hence we have the corresponding central extension $1 \to \pi_1(G)_C \to \widetilde{G}^{\vee}_{\theta} \to G^{\vee}_{\theta\,\mathrm{top}} \to 1$ of twisted topological groups. An easy topological consideration show that the boundary map (first Chern class) $H^1(C, G^{\vee}_{\theta\,\mathrm{top}}) \to H^2(C, \pi_1(G)) = \Gamma/\Gamma_{\mathrm{root}}$ is bijection. Since the space of holomorphic structures ($= \bar{\partial}$-connections) on a given topological G'-bundle is nonempty and connected, we get the desired identification of the set of connected components of the stack $G^{\vee}_{\theta}$-tors with $\Gamma/\Gamma_{\mathrm{root}}$.

2. Let Lie $\mathcal{H}_\theta$ be the Lie algebra of $\mathcal{H}_\theta$ (which is a vector bundle on C) and $\exp$: Lie $\mathcal{H}_\theta \to \mathcal{H}_\theta$ be the exponential map. On the open set $U := C \backslash R_\theta$ the map exp is surjective, and ker exp is a local system $\Gamma_{\widetilde{U}}(1)$, which is $\Gamma(1)$ twisted by W-torsor $\widetilde{U} := p^{-1}(U) \to U$ (here W acts on Γ in a standard way). Let $j_{U_*}\Gamma_{\tilde{U}}(1)$ be the direct image extension of $\Gamma_{\tilde{U}}(1)$ to C (here $j_U : U \hookrightarrow C$). Then ker exp $= j_{U_*}\Gamma_{\tilde{U}}(1)$, and cokerexp $= \bigoplus_{x \in \mathcal{R}_{\theta 1}} \mathbb{Z}/2x$ by 5.2.1 (v).

   Since $H^i(C, \mathrm{Lie}\ \mathcal{H}_\theta)$ are $\mathbb{C}$-vector spaces and $H^2(C, \mathrm{Lie}\ \mathcal{H}_\theta) = 0$ the group of connected components of the stack of $\mathcal{H}_\theta$-torsors is equal to hyper-cohomology group $H^2(c, \mathcal{F}^{\cdot})$, where $\mathcal{F}^{\cdot}$ is a constructible complex $\mathcal{F}^0 := \mathrm{Lie}\ \mathcal{H}_\theta \overset{\exp}{\to} \mathcal{F}^1 := \mathcal{H}_\theta$. Therefore $H^2(C, \mathcal{F}^{\cdot}) = \mathrm{coker}(\mathbb{Z}/2^{\mathcal{R}_{\theta 1}} \overset{\partial}{\to}$

$H^2(C, j_{U*}\Gamma_{\tilde{U}}(1))$. But $H^2(C, j_{U*}\Gamma_{\tilde{U}}(1)) = \Gamma_W$, and an easy local computation shows that for $x \in \mathcal{R}_{\theta 1}$ the morphism $\partial : \mathbb{Z}/2_x \to \Gamma_W$ is given by formula $\partial(1) = \gamma_x$. Since obviously both type 2 and type 3 roots γ have zero classes in Γ_W and any type 1 root occurs as some γ_x by 6.1(iii), we see that $H^2(C, \mathcal{F}^{\cdot}) = \Gamma/\Gamma_{\mathrm{root}}$.

3. We identified canonically the set of connected components of both $\mathcal{H}_\theta$-tors and G'_C-tors with $\Gamma/\Gamma_{\mathrm{root}}$. It is easy to see that the map induced by φ-induction i_{tors} is the identical map of $\Gamma/\Gamma_{\mathrm{root}}$. We are done.

6.4. The functor φ defines the induction functors $\mathcal{H}_\theta$-tors $\to$ $(p_* H_{\tilde{C}_\theta})^W$-tors $\to$ $H_{\tilde{C}_\theta}$-tors. Here $H_{\tilde{C}_\theta,W}$-tors denotes the category of W-equivariant H-torsors on $\tilde{C}_\theta$. Let us compare these categories.

Take $\mathcal{F} \in H_{\tilde{C}_\theta,W}$-tors. For a point $x \in \tilde{C}_\theta$ the fiber $\mathcal{F}_x$ is a W_x-equivariant H-torsor; let $c\ell_2\mathcal{F} := c\ell\mathcal{F}_x \in H^1(W_x, H)$ be its class. If $W_x \neq \{1\}$ then $W_x = \{1, \sigma_\gamma\}$ and $H^1(W_x, H) = H^{\gamma -}/H^{\gamma - 0}$, where $H^{\gamma -} := \{h \in H : \sigma_\gamma h = h^{-1}\}$, $H^{\gamma - 0} := \{h \in H : h = \sigma_\gamma(\ell) \cdot \ell^{-1}\}$ = connected component of $H^{\gamma -}$.

We will say that $\mathcal{F}$ is pointwise trivial if $c\ell_x\mathcal{F} = 0$ or, equivalently, $\mathcal{F}_x^{W_x} \neq$) for any $x \in \tilde{C}_\theta$. Denote by $H_{\tilde{C}_\theta,W}$-tors$_0$ the full subcategory of such $\mathcal{F}$'s. It is easy to see that for any $\mathcal{T} \in (p_* H_{\tilde{C}_\theta}^W$-tors the corresponding $H_{\tilde{C}_\theta,W}$-torsor is pointwise trivial.

For $\mathcal{F} \in H_{\tilde{C}_\theta,W}$-tors$_0$ and a type 2 point x the fiber $\mathcal{F}_x^{W_x}$ has 2 connected components. A+-structure on $\mathcal{F}$ is a choice for any type 2 point x of a component $\mathcal{F}_x^+ \subset \mathcal{F}_x^{W_x}$ such that for any $w \in W$ one has $w(\mathcal{F}_x^+) = \mathcal{F}_{wx}^+$. Denote by $\mathcal{F}^+ \subset \mathcal{F}$ a subsheaf of sections that take value in $\mathcal{F}_x^+$ for any type 2 point x. The pointwise trivial torsors with +-structure form a category

$H_{\tilde{C}_\theta,W}$-tors$_0^+$. If $\mathcal{T}$ is an $\mathcal{H}_\theta$-torsor, then the corresponding $H_{\tilde{C}_\theta}$-torsor $\mathcal{F}$ carries a natural $+$-structure $\mathcal{F}_x^+ := \varphi(\mathcal{T}_{p(x)})$, hence the functor $\mathcal{H}_\theta$-tors $\to$ $H_{\tilde{C}_\theta,W}$-tors$_0^+$.

6.4.1 Lemma. *The functors* $\mathcal{H}_\theta$-tors $\to$ $H_{\tilde{C}_\theta,W}$-tors$_0^+$, $p_*(H_{\tilde{C}_\theta})^W$-tors $\to$ $H_{\tilde{C}_\theta,W}$-tors$_0$ *are equivalence of categories.*

Proof: Easy. The inverse functors are respectively $\mathcal{F} \mapsto (p_*\mathcal{F}^+)^W$, $\mathcal{F} \mapsto (p_*\mathcal{F})^W$. $\square$

Denote by $|?{-}tors|$the group of isomorphism classes of corresponding torsors. Consider the forgetting of W-action functor $0 : H_{\tilde{C}_\theta,W}$-tors$_0 \to H_{\tilde{C}_\theta}$-tors.

6.4.2 Lemma. *The corresponding morphism of groups* $0 : |H_{\tilde{C}_\theta,W}$-tors$| \to |H_{\tilde{C}_\theta}$-tors$| = \mathrm{Pic}(\tilde{C}_\theta) \otimes \Gamma$ *is injective.*

Proof: The isomorphism classes of $H_{\tilde{C}_\theta}$-torsors trivial as $H_{\tilde{C}_\theta}$-torsors form a group $H^1(W, H)$. The pointwise trivial ones form a subgroup

$$\begin{aligned} H^1(W,H)_0 :=& \bigcap_{x\in\tilde{C}_\theta} \ker(H^1(W,H) \to H^1(W_x,H)) \\ =& \bigcap_{\gamma\in S} \ker(H^1(W,H) \to H^1(W_\gamma,H)) \end{aligned}$$

(see 6.1(iii). To see that $H^1(W,H)_0 = 0$ consider the short exact sequence

$$1 \to H^W \to H \xrightarrow{\nu} \prod_{\gamma\in S} H^{\gamma-0} \to 1,$$

$$\nu(h) := (\sigma_\gamma(h)\cdot h^{-1})_{\gamma\in S} = (i_\gamma\chi_\gamma(h^{-1})_{\gamma\in S}$$

(see 5.2.1). If $\alpha \in Z^1(W,H)$ is a cocycle with a class in $H^1(W,H)_0$, then $\alpha(\sigma_\gamma) \in H^{\gamma-0}$ (since $H^1(W_\gamma,H) = H^{\gamma-}/H^{\gamma-0}$). Hence $(\alpha(\sigma_\gamma))_{\gamma\in S} \in \nu(H)$, i.e., for some $h \in H$ one has $\alpha(\sigma_\gamma) = \sigma_\gamma(h) \cdot h^{-1}$ for any $\gamma \in S$. Since σ_γ, $\gamma \in S$, generate W we see that $\alpha(w) = w(h) \cdot h^{-1}$ for any w, i.e., α is hololologous to 0. $\square$

Let indices 0 denote the connected component of an algebraic group.

6.4.3 Corollary. *One has the isomorphism* $|H_{\tilde{C}_\theta,W}\text{-tors}|^0 \underset{\sim}{\to} (\mathrm{Pic}(\tilde{C}_\theta) \otimes \Gamma)^{W_0}$. *The corresponding map* $|\mathcal{H}_\theta\text{-tors}|^0 \to |\mathrm{Pic}(\tilde{C}_\theta) \otimes \Gamma|^W$ *is an isogenic with kernel a 2-group.*

Proof: The second statement follows from the fact that the group of $+$-structures on a trivial $H_{\tilde{C}_\theta,W}$-torsor coincides with $\prod_{\gamma \in S} \mathbb{Z}/2^{\mathcal{R}_\gamma}/\delta(\mathbb{Z}/2)$, where $\mathcal{R}_\gamma := \{x \in \mathcal{R}_\theta : \gamma_x = \gamma\}$, and $\delta : \mathbb{Z}/2 \to \mathbb{Z}/2^{\mathcal{R}_\gamma}$ is diagonal embedding. $\square$

Appendix A
Rings of Twisted Differential Operators

A1. Basic Definitions and Equivalences

In this section we will give several descriptions of category of twisted differential operator rings. Below X is a smooth algebraic or analytic variety over $\mathbb{C}$.

Definition. *Let D be a sheaf of rings on X equipped with a ring filtration $D_0 \subset D_1 \subset D_2 \subset \cdots$ (we have $D_i \cdot D_j \subset D_{i+j}$) and a ring isomorphism $D_0 = \mathcal{O}_X$. We call D a ring of twisted differential operators (or simply a tdo) if*

(i) *The graded ring is a commutative $\mathcal{O}_X$-algebra (with respect to $\mathcal{O}_X = D_0 \hookrightarrow \mathrm{gr}.D$) such that the corresponding morphism $S^{\cdot}(D_1/D_0) \to \mathrm{gr}.D$ is isomorphism.*

(ii) *ThePoisson bracket $\{\ ,\ \} : \mathrm{gr}_a D \times \mathrm{gr}_b D \to gr_{a+b} D$ (defined by formula $\{f, g\} := \tilde{f}\tilde{g} - \tilde{g}\tilde{f} \mod D_{a+b-2}$ where $\tilde{f} \in D_a$, $\tilde{g} \in D_b$ are representatives of f, g) defines the isomorphism $\sigma : D_1/D_0 \underset{\sim}{\to} \mathcal{T}_X$, $\sigma(\tau)(f) = \{\tau, f\}$.* $\square$

Note that for a tdo D the filtration $D.$ is completely determined by $\mathcal{O}_X = D_0 \hookrightarrow D$: one has $D_1 = \{\partial \in D : [\partial, D_0] \subset D_0\}$, $D_i = D_1^i$.

A1.2 Example: If $\mathcal{L}$ is a line bundle on X, then $D_{\mathcal{L}} :=$ ring of differential operators acting on $\mathcal{L}$ is a tdo (with $d_{\mathcal{L}} :=$ operators of order $\leq i$).

Clearly tdo's on X form a category (a groupoid) $TDO(X)$. Below we will give several descriptions of this groupoid.

A1.3. Let $\widetilde{\mathcal{T}}$ be a sheaf of $\mathcal{O}_X$-modules equipped with a Lie algebra structure $[\ \ ]$, a section 1 of the center of $\widetilde{\mathcal{T}}$, and an $\mathcal{O}_X$-linear map $\sigma : \widetilde{\mathcal{T}} \to \mathcal{T}_X$ such that the sequence $0 \to \mathcal{O}_X \xrightarrow{i} \widetilde{\mathcal{T}} \xrightarrow{\sigma} \mathcal{T}_X \to 0$, where $i(f) := f \cdot 1$ is exact and one has $[\partial_1, f\partial_2] = \sigma(\partial_1)(f)\partial_2 + f[\partial_1, \partial_2]$ for $\partial_1, \partial_2 \in \widetilde{\mathcal{T}}$, $f \in \mathcal{O}_X$. Clearly ∂ is a Lie algebra map, i identifies $\mathcal{O}_X$ with an abelian ideal of $\widetilde{\mathcal{T}}$ and adjoint action of $\widetilde{\mathcal{T}}$ on $\mathcal{O}_X$ with σ.

We will call such $\widetilde{\mathcal{T}}$ an $\mathcal{O}$-extension of $\mathcal{T}_X$. These form a groupoid $TDO'(X)$. Note that $TDO'(X)$ is a "$\mathbb{C}$-vector space in categories": We can form $\mathbb{C}$-linear combinations of $\mathcal{O}$-extensions (Baer sum construction).

A1.4 Lemma. *The groupoids TDO(X) and $\mathcal{T}DO'(X)$ are canonically equivalent.*

Proof: The corresponding mutually inverse function $\mathcal{T}DO(X) \rightleftarrows \mathcal{T}DO'(X)$ are the following ones. If D is a tdo, then $\widetilde{\mathcal{T}}_D := D_1$ is an $\mathcal{O}$-extension of $\mathcal{T}_X$ (the $\mathcal{O}_X$-module structure on $\widetilde{\mathcal{T}}_D$ comes from left multiplication by functions. Conversely, if $\mathcal{T}$ is an $\mathcal{O}$-extension, then let $D_{\widetilde{\mathcal{T}}}$ be an associative algebra generated by $\widetilde{\mathcal{T}}$ with the only relations $\partial_1 \cdot \partial_2 - \partial_2 \cdot \partial_1 = [\partial_1, \partial_2]$, $f_1 \cdot f_2 = f_1 f_2$, $1 = 1 \in \widetilde{\mathcal{T}}$, $f \cdot \partial = f\partial$, for $\partial_i \in \widetilde{\mathcal{T}}$, $f_i \in \mathcal{O}_X \subset \widetilde{\mathcal{T}}$ (here $\cdot$ denotes the product in D). This $D_{\widetilde{\mathcal{T}}}$ is the tdo that corresponds to $\widetilde{\mathcal{T}}$. $\square$

A1.5. Let $d : A^n \to A^{n+1}$ be a morphism of sheaves of abelian groups on X, considered as length 2 complex $A^{\cdot}$ supported in degrees n and $n+1$. An $A^{\cdot}$-torsor is a pair $(\mathcal{F}, c)$, where $\mathcal{F}$ is an A^n-torsor and $c : \mathcal{F} \to A^{n+1}$ is a map such that $c(a + \varphi) = d(a) + c(\varphi)$ for $a \in A^n$, $\varphi \in \mathcal{F}$ (in other words, curv is a trivialization of the induced A^{n+1}-torsor $d(\mathcal{F})$). These $A^{\cdot}$-torsors form a groupoid $A^{\cdot}$-tors. One has $\mathrm{Aut}\ \mathcal{F} = \Gamma(X, \ker\ d) = H^n(X, A^{\cdot})$, and isomorphism classes of $A^{\cdot}$-torsors are in a natural 1-1 correspondence with $H^{n+1}(X, A^{\cdot})$.

Remark. $A^{\cdot}$-tors is a stack in Picard categories on X; if $A^{\cdot}$ is a complex of $\mathbb{C}$-vector spaces, thern $A^{\cdot}$-tors is a $\mathbb{C}$-vector space in categories (one forms $\mathbb{C}$-linear combinations of torsors in an obvious way). If d is surjective, then $A^{\cdot}$tors = (ker d)-tors.

Consider the truncated de Rham complex $\Omega_X^{\geq 1} := (\Omega_X^1 \to \Omega^{2c\ell})$, where $\Omega^{2c\ell}$ are closed 2-forms.

A1.6 Lemma. *One has a canonical equivalence of* $\mathbb{C}$*-vector space in categories* $C : \mathcal{T}DO'(X) \underset{\sim}{\to} \Omega_X^{\geq 1}$-tors.

Proof: Let $\widetilde{\mathcal{T}}$ be an $\mathcal{O}$-extension of $\mathcal{T}_X$. Connections ∇ on $\widetilde{\mathcal{T}}$ form an Ω_X^1-torsor $C(\widetilde{\mathcal{T}})$ (for a connection ∇ and a 1-form ν one has $(\nu+\nabla)(\tau) := \nu(\tau) + \nabla(\tau), \tau \in \mathcal{T}_X$). A curvature of ∇ is a closed 2-form curv(∇) defined by formula $\mathrm{curv}(\nabla)(\tau_1 \wedge \tau_2) := [\nabla(\tau_1), \nabla(\tau_2)] - \nabla([\tau_1, \tau_2])$; one has $\mathrm{curv}(\nu + \nabla) = d\nu + \mathrm{curv}(\nabla)$. So our functor C is $\widetilde{\mathcal{T}} \mapsto (\mathcal{C}(\widetilde{\mathcal{T}}), \mathrm{curv})$. Obviously this is a $\mathbb{C}$-linear equivalence of categories. $\square$

By A1.6 we may identify the set of isomorphism classes of tdo's with $H^2(X, \Omega_X^{\geq 1})$. For a tdo D we will denote by $c_1(D) \in H^2(X, \Omega_X^{\geq 1})$ the corresponding class.

A1.7. For a tdo D a connection ∇ on D is a connection on a corresponding $\mathcal{O}$-extension of $\mathcal{T}_X$. Note that pairs (D, ∇), ∇ is a connection on a tdo D, are rigid: the only automorphism of D that preserves ∇ is identity. The pairs

(D, ∇) are in 1-1 correspondence with closed 2-forms; for $\omega \in \Omega^{2v\ell}(X)$ we will denote by $(D_\omega, \nabla_\omega)$ a unique *vp* to a canonical isomorphism) tdo with curve $(\nabla) = \omega$. A corresponding $\Omega_X^{\geq 1}$-torsor $(\mathcal{F}_\omega, \mathrm{curv}_\omega)$ is given by formula $\mathcal{F}_\omega, \mathrm{curv}_\omega;)$ is given by formula $\mathcal{F}_\omega = \Omega_X^1$, $\mathrm{curv}_\omega(\nu) = d\nu + \omega$. $\square$

Now consider a cotangent bundle $T^* = T^*(X) \xrightarrow{\pi} X$. This is a vector bundle over X; also T^* carries a canonical symplectic 2-form ω such that π is a polarization. If ν is a 1-form on X, and $t_\nu : T^* \to T^*$, $t_\nu(a) = a + \nu^{\cdot}_{\pi(a)}$, is translated by ν, then $t_\nu^*(\omega) = \pi^*(d\nu) + \omega$.

A1.8 Definition. *A twisted cotangent bundle is a T^*-torsor $\phi \xrightarrow{\pi_\phi} X$ (i.e., π_ϕ is a fibration equipped with a simple transitive action of T^* along the fibers) together with a symplectic form ω_ϕ on ϕ such that π_ϕ is a polarization for ω_ϕ, and for any 1-form ν one has $t_\nu^*(\omega_\phi) = \pi_\phi^* d\nu + \omega$.* $\square$

For a twisted cotangent bundle ϕ we will denote by A_ϕ the $\mathcal{O}_X$-algebra $\pi_{\phi^*}\mathcal{O}_\phi$. Then A_ϕ carries Poisson bracket $\{\ ,\ \}$ (defined by ω_ϕ) and a filtration $A_{\phi_i} =$ functions of degree $\leq i$ along the fibers of π_ϕ. Clearly one has $A_{\phi_i} = \{\varphi \in A)\phi : \{\varphi, \mathcal{O}_X\} \subset A_{\phi_{i-1}}\} = S^i A_{\phi_1}$, and the graded algebra of gr.A_ϕ coincides with $A_{T^*} = S^{\cdot}\mathcal{T}_X$.

A1.9 Remarks: (i) The T^*-torsor structure on ϕ is uniquely determined by the symplectic structure ω_ϕ and the polarization π_ϕ (since the infinitesimal action of a 1-form $\nu \in \Omega^1(X)$ is given by a vector field ξ_ν, $\xi_\nu \omega_\phi = \pi_\phi^*(\nu)$).

(ii) Twisted cotangent bundles over X for a groupoid $\mathcal{TCB}(X)$. According to (i), $\mathcal{TCB}(X)$ is a full subcategory of the category of triples (Y, ω_Y, π_Y) where (Y, ω_Y) is a symplectic manifold and $\pi_Y : Y \to X$ is a polarization (for the symplectic structure).

A1.10 Lemma. *One has a canonical equivalence of categories* $\Gamma : \mathcal{TCB}(X) \underset{\sim}{\rightarrow} \Omega_X^{\geq 1}$*-torsor.*

Proof: Put $\Gamma(\phi) = \Omega^1$-torsor of section of ϕ; the map curv: $\Gamma(\phi) \to \Omega_X^{2c\ell}$ is $\mathrm{curv}(\gamma) := \gamma^*(\omega_\phi)$. Note that the corresponding $\mathcal{O}_X$-extension $\widetilde{\mathcal{T}}_\phi$ of $\mathcal{T}_X$ is A_{ϕ^1} equipped with the bracket $\{\ ,\ \}$.

The inverse functor Γ^{-1} maps $\Omega_X^{\geq 1}$-torsor $(\mathcal{F}, \mathrm{curv})$ to $(\phi, \pi_\phi, \omega_\phi)$, where $\pi_\phi : \phi \to X$ is the space of torsor $\mathcal{F}$, and the symplectic form ω_ϕ is a unique form such that for a section $\gamma \in \mathcal{F}$ of π_ϕ the corresponding isomorphism $T^*X \underset{\sim}{\rightarrow} \phi$, $0 \mapsto \gamma$, identifies ω_ϕ with $\omega + \pi_p^* \,\mathrm{curv}(\gamma)$.

A1.11. Let D be a tdo, and ϕ be the corresponding twisted cotangent bundle. Then D is a "canonical quantization" of ϕ in a sense that D is a deformation of a commutative algebra, A_ϕ. Precisely, one has a canonical family $\mathbb{D} = \{D_t\}$ of sheaves of filtered rings on X parameterized by $t \in \mathbb{P}^!$ (i.e., $\mathbb{D}$ is a flat $\mathcal{O}_{\mathbb{P}^1}$-algebra) such that

(i) for $t \neq \infty$ one has $D_t = D_{t\widetilde{\mathcal{T}}}$ (here $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_D$; for a product of an $\mathcal{O}_X$-extension by $t \in \mathbb{C}$; see 2.2). In particular, $D_1 = D$, $D_0 = D_{\mathcal{O}_X}$.

(ii) $D_\infty = A_\phi$, and the ω_ϕ-Poisson bracket on A_ϕ is given by usual formula $\{\varphi_1, \varphi_2\} = [t(\tilde{\varphi}_1, \tilde{\varphi}_2 - \tilde{\varphi}_2\tilde{\varphi}_1)] \bmod t^{-1}$ (here $\varphi_i \in D_\infty$, and $\tilde{\varphi}_i$ are any sections of $\mathbb{D}$ round $t = \infty$ such that $\tilde{\varphi}_i(\infty) = \varphi_i$).

(iii) $\mathrm{gr}_a \mathbb{D} = (S^a \mathcal{T}_X)(-a)$.

Here is a construction of $\mathbb{D}$. Define first the restriction $\mathbb{D}|_{\mathbb{P}^1\setminus\{\infty\}}$. The ring $\mathbb{D}(\mathbb{P}^1\setminus\{\infty\})$ of sections is a $\mathbb{C}[t]$-algebra generated by subalgebra $\mathcal{O}_X$ and a subsheaf $\widetilde{\mathcal{T}}$ with the only relations $\partial_1 \cdot \partial_2 - \partial_2 \cdot \partial_1 = [\partial_1, \partial_2]$, $f \cdot \partial = f\partial$, $tf = [f]$; here ∂_i, $\partial \in \widetilde{\mathcal{T}}$, $f \in \mathcal{O}_X$ and $[f] \in \widetilde{\mathcal{T}}$ is f considered as element of $\mathcal{O}_X \subset \widetilde{\mathcal{T}}$. Let $j : \mathbb{P}^1 \backslash \{\infty\} \hookrightarrow \mathbb{P}^1$ be embedding. Our $\mathbb{D}$ is a subalgebra of $j_* \mathbb{D}_{\mathbb{P}^1 \backslash \{\infty\}}$ generated by $\mathcal{O}_X$ and $\mathcal{O}_{\mathbb{P}^1}(-\infty) \cdot \widetilde{\mathcal{T}}$. The identification $A_\phi \underset{\sim}{\to} D_\infty$ assigns to $\partial \in \widetilde{\mathcal{T}} \subset A_\phi$ the element $(t^{-1}\partial)_\infty \in D_\infty = \mathbb{D}/t^{-1}\mathbb{D}$.

A1.12. Let us see what the above constructions mean in case $D = D_{\mathcal{L}}$, $\mathcal{L}$ is a line bundle. The corresponding $\mathcal{O}_X$-extension $\widetilde{\mathcal{T}}_{\mathcal{L}} = \widetilde{\mathcal{T}}_{D_{\mathcal{L}}}$ consists of pairs $(\tau, \tilde{\tau})$, where τ is a vector field and $\tilde{\tau}$ is an action of τ on $\mathcal{L}$. The $\Omega_X^{\geq 1}$-torsor $(\mathcal{F}_{\mathcal{L}}, \mathrm{curv}_{\mathcal{L}}) := C(\widetilde{\mathcal{T}}_{\mathcal{L}})$ is the sheaf of connection on $\mathcal{L}$, $\mathrm{curv}_{\mathcal{L}}$ is a usual curvature. Note that this functor $\mathcal{O}_X^*$-tors $\to \Omega_X^{\geq 1}$-tors is precisely the push-out functor for the morphism $d\log : \mathcal{O}_X^* \to \Omega_X^{1c\ell} (\subset \Omega_X^{\geq 1}[1])$. In particular it transforms $\otimes$ to the sum of torsors. For any $\lambda \in \mathbb{C}$ we put $D_{\mathcal{L}^\lambda} := \lambda D_{\mathcal{L}}$. One has $c_1(D_{\mathcal{L}}) = c_1(\mathcal{L}) \in H^2(X, \Omega_X^{\geq 1}$.

A1.13. A tdo D is called locally trivial if locally it is isomorphic to $D_X = D_{\mathcal{O}_X}$; according to A1.6 the locally trivial tdo's are the same as $\Omega_X^{1c\ell}$-torsors. Note that in analytic situation each tdo is locally trivial. In algebraic situation this is not true in general. For example, let X be a compact algebraic variety. The space of isomorphism classes of tdo's $H^2(X, \Omega_X^{\geq 1})$ coincides with Hodge filtration space $F^1 H^2_{DR}$, and it is easy to see that the locally trivial ones correspond precisely to a $\mathbb{C}$-linear combinations of an algebraic cycles classes.

A1.14 Definition. *Let D be a tdo. A D-module M is lisse if M is coherent as $\mathcal{O}_X$-module.* $\square$

A1.15 Lemma. *Let D be a tdo, and M be a non-zero lisse D-module then*

(i) *M is a vector bundle of dimension, say, d.*

(ii) *One has a canonical isomorphism of tdo's $D \xrightarrow{\sim} D_{\det M)^{1/d}}$. In particular, D is locally trivial.*

Proof. (i) is well known (see, e.g., [Bo]). The isomorphism $D \xrightarrow{\sim} D_{(\det M)^{1/d}}$ comes from the isomorphism of $\mathcal{O}$-extensions $d_M \widetilde{\mathcal{T}}_D \xrightarrow{\sim} \widetilde{\mathcal{T}}_{\det M}$: an element $\tilde{\tau} \in \mathcal{T}_D$ acts on $\det M = \Lambda^d M$ by Leibnitz rule $\tilde{\tau}(m_1 \wedge \ldots \wedge m_d) = \tilde{\tau}(m_1) \wedge \ldots \wedge m_d + \ldots + m_1 \wedge \ldots \wedge \tilde{\tau}(m_d)$ □

One has a following relation between twisted D-module structures and integrable projective connections. Let $\mathcal{E}$ be a quasicoherent $\mathcal{O}_X$-module. An action of a vector field $\tau \in \mathcal{T}_X$ on $\mathcal{E}$ is an endomorphism $\tilde{\tau} \in \mathrm{End}_{\mathbb{C}}\mathcal{E}$ such that for $f \in \mathcal{O}_X$, $e \in \mathcal{E}$ one has $\tilde{\tau} f e = f \tilde{\tau} e + \tau(f) e$. Let $\widetilde{\mathcal{T}}_{\mathcal{E}}$ be the sheaf of all such pairs $(\tau, \tilde{\tau})$: this is an $\mathcal{O}_X$-module and Lie algebra in an obvious manner; we have an exact sequence $0 \to \mathrm{End}_m \mathcal{E} \to \widetilde{\mathcal{T}}_{\mathcal{E}} \xrightarrow{\sigma} \mathcal{T}_X$ of Lie algebras. Clearly $\mathcal{O}_X \cdot id_{\mathcal{E}} \subset \mathrm{End}_{\mathcal{O}_X} \mathcal{E} \subset \widetilde{\mathcal{T}}_{\mathcal{E}}$ are ideals; put $\overline{\mathrm{End}}\ \mathcal{E} := \mathrm{End}\ \mathcal{E} / \mathcal{O}_X \cdot Id_{\mathcal{E}}$, $\widetilde{\mathcal{T}}_{\mathcal{E}} := \widetilde{\mathcal{T}}_{\mathcal{E}} / \mathcal{O}_X \cdot id_{\mathcal{E}} \xrightarrow{\sigma} \mathcal{T}_S$.

A1.16 Definition. (i) *A projective connection $\mathcal{E}$ is an $\mathcal{O}_X$-linear section $\overline{\nabla} : \mathcal{T}_X \to \overline{\mathcal{T}}_{\mathcal{E}}$ of $\bar{\sigma}$. Such $\overline{\nabla}$ is integrable if it commutes with brackets.*

(ii) *Let D be a tdo. A D-structure on $\mathcal{E}$ is an action of D on $\mathcal{E}$ that extends the given $\mathcal{O}_X$-action.*

Clearly a D-structure on $\mathcal{E}$ is the same as an $\mathcal{O}_X$-linear morphism of Lie algebras $\alpha : \widetilde{\mathcal{T}}_D \to \widetilde{\mathcal{T}}_{\mathcal{E}}$ such that $\sigma\alpha = \sigma$ and $\alpha(1) = id_{\mathcal{E}}$. Such α defines an integrable projective connection $\overline{\nabla}_\alpha$ on $\mathcal{E}$ by formula $\overline{\nabla}_\alpha(\tau) = \alpha(\tilde{\tau})$ mod

$\mathcal{O}_X id_{\mathcal{E}}$, where $\tilde{\tau} \in \mathcal{T}$, $\sigma(\tilde{\tau} = \tau)$.

A1.17 Lemma. *Assume that the map $\mathcal{O}_X \to \operatorname{End} \mathcal{E}$, $f \mapsto f\ id_{\mathcal{E}}$, is injective. Then the above map $\alpha \mapsto \overline{\nabla}_\alpha$ from the set of pairs (D, α), D is a tdo, α is a D-structure on $\mathcal{E}$, to the set of projective integrable connections on $\mathcal{E}$ is bijective.*

Proof: One constructs the inverse map as follows. Let $\overline{\nabla} : \mathcal{T}_X \to \overline{\mathcal{T}}_{\mathcal{E}}$ be an integrable projective connection. Then $\widetilde{\mathcal{T}}_{\overline{\nabla}} := \mathcal{T}_X \ \times_{\widetilde{\mathcal{T}}_{\mathcal{E}}} \widetilde{\mathcal{T}}_{\mathcal{E}}$ is an $\mathcal{O}$-extension of $\mathcal{T}_X$, and the projection $\alpha_{\overline{\nabla}} : \widetilde{\mathcal{T}}_{\overline{\nabla}} \to \widetilde{\mathcal{T}}_{\mathcal{E}}$ defines the $D_{\widetilde{\mathcal{T}}_{\overline{\nabla}}}$-structure on $\mathcal{E}$. $\square$

A2 Subprincipal Symbols

Let $\Omega = \det\ \Omega^1_X$ be the sheaf of volume forms on X, and $\widetilde{\mathcal{T}}_\Omega$ be the corresponding $\mathcal{O}$-extension of $\mathcal{T}_X$. One has a canonical section $\ell : \mathcal{T}_X \to \widetilde{\mathcal{T}}_\omega$ which assigns to $\partial \in \mathcal{T}_X$ its Lie derivative $\ell(\partial)$. Clearly ℓ commutes with bracket and for $f \in \mathcal{O}_X$ one has $f\ell(\partial) = \ell(f\partial) - \partial(f)$.

A2.1. Now let $\widetilde{\mathcal{T}}$ be any $\mathcal{O}$-extension of $\mathcal{T}_X$. Denote by $\widetilde{\mathcal{T}}^0$ and $\mathcal{O}$-extension of $\mathcal{T}_X$ together with isomorphism of sheaves $* : \mathcal{T} \underset{\sim}{\to} \widetilde{\mathcal{T}}^0$ such that $*[\tau_1, \tau_2] = -[*\tau_1, *\tau_2]$, $*(f\tau) = f{*\tau} + \tau(f)$, $\sigma(*\tau) = -\sigma(\tau)$, $*(1) = 1$ for $\tau_i \in \widetilde{\mathcal{T}}$, $f \in \mathcal{O}_X$. Clearly $*$ extends to isomorphism of tdo's $* : D^0_{\widetilde{\mathcal{T}}} \to D_{\widetilde{\mathcal{T}}^0}$, where $D_{\widetilde{\mathcal{T}}^0}$ means the ring $D_{\widetilde{\mathcal{T}}}$ with reversed multiplication. Note that $(\widetilde{\mathcal{T}}^0)^0$ and $** = id$. Denote by $\widetilde{\mathcal{T}}^{01}$ the Baer difference $\widetilde{\mathcal{T}}_\Omega - \widetilde{\mathcal{T}}$ of $\mathcal{O}$-extensions (see A1.3), so an element of $\widetilde{\mathcal{T}}^{01}$ is a pair (a, b) $a \in \widetilde{\mathcal{T}}_\Omega$, $b \in \widetilde{\mathcal{T}}$, such that $\sigma(a) = \sigma(b)$, modulo relations $(a, b) = (a + f, b + f)$, $f \in \mathcal{O}_X$. One has a canonical isomorphism

$\widetilde{\mathcal{T}}^0 \xrightarrow{\sim} \widetilde{\mathcal{T}}^{01}$ defined by formula $*\tau \mapsto (-\ell\sigma(\tau), -\tau)$, $\tau \in \widetilde{\mathcal{T}}$, hence we have $* : D^0_{\widetilde{\mathcal{T}}} \underset{\sim}{\rightarrow} D_{\widetilde{\mathcal{T}}^0} = D_{\widetilde{\mathcal{T}}^{01}}$.

A2.2. Consider the $\mathcal{O}$-extension $\widetilde{\mathcal{T}}_{\Omega^{1/2}}$ and the corresponding tdo $D_{\Omega^{1/2}}$. Since $\widetilde{\mathcal{T}}^{01}_{\Omega^{1/2}} = \widetilde{\mathcal{T}}_{\Omega^{1/2}}$ we have $* : D^0_{\Omega^{1/2}} \to D_{\Omega^{1/2}}$, i.e., * is automorphism of the sheaf $D_{\Omega^{1/2}}$ such that $*(\partial_1\partial_2) = *(\partial_2) * (\partial_1)$. $*^2 = id$ and $*$ induces multiplication by ℓ^i on $D_{\Omega^{1.2}i}/D_{\Omega^{1.2}i-1} = S^i\mathcal{T}_X$. Denote by $D^{\pm}_{\Omega^{1/2}}$ the ±1-eigenspaces of * on $D_{\Omega^{1/2}}$, so $D_{\Omega^{1/2}} = D^+_{\Omega^{1/2}} \oplus D^-_{\Omega^{1/2}}$. Note that gr $D^+_{\Omega^{1/2}} = \oplus S^{2i}\mathcal{T}_X$, gr $D^-_{\Omega^{1/2}} = \oplus S^{2i+1}\mathcal{T}_X$, and the $\pm$-grading is compatible with bracket: for a $\pm$-homogeneous elements $a, b \in D_{\Omega^{1/2}}$ the elements $[ab] = ab - ba$ is also homogeneous.

A2.3. Let D be a tdo. Put $\widetilde{\mathrm{gr}}_a D := D_a/D_{a-2}$. We will consider $\widetilde{\mathrm{gr}}.D = \oplus\, \widetilde{gr}_a D$ as a Lie algebra with bracket $\{\ ,\ \} : \widetilde{\mathrm{gr}}_a D \times \widetilde{\mathrm{gr}}_b D \to \widetilde{\mathrm{gr}}_{a+b-1} D$ that comes from the bracket $[\ ,\ ]$ on D. So $S^{\cdot}\mathcal{T}_X = \mathrm{gr}.D$ equipped with a usual Poisson bracket is a quotient of $\widetilde{\mathrm{gr}}.D$ modulo the abelian ideal.

A2.4 Example: The $\pm$-grading on $D_{\Omega^{1/2}}$ induces a canonical isomorphism $\widetilde{gr}_a D_{\Omega^{1/2}} = S^a\mathcal{T}_X \oplus S^{a-1}\mathcal{T}_X$ which identifies $\{\ \}$ with the usual Poisson bracket.

This example could be generalized as follows. For any tdo D consider an $\mathcal{O}$-extension $\widetilde{\mathcal{T}}^{\vee} := \widetilde{\mathcal{T}}_D - \widetilde{\mathcal{T}}_{\Omega^{1/2}}$. Let $(\phi, \pi_\phi, \omega_\phi)$ be its twisted cotangent bundle, and $A. = \pi_{\phi*}\mathcal{O}_\phi$, be the corresponding filtered commutative algebra with Poisson bracket $\{\ \}$, so $A_n - S^n(\widetilde{\mathcal{T}}^{\vee})$ (see A1.8, A1.10). Put $\widetilde{\mathrm{gr}}.A = A./A._{-2}$: this is a commutative algebra, and $\{\ ,\ \}$ induces the Lie algebra structure on $\widetilde{\mathrm{gr}}.A$.

A2.5 *One has a canonical isomorphism $\tilde{\sigma} : \widetilde{gr}.D \underset{\sim}{\rightarrow} \widetilde{gr}.A$, compatible with brackets, that lifts the isomorphism $\sigma. : gr.D \underset{\sim}{\rightarrow} gr.A = S^{\cdot}\mathcal{T}_X$.*

Proof: Let us construct the inverse isomorphism $\alpha : \widetilde{\mathrm{gr}}.A \ usr \ \widetilde{\mathrm{gr}}.D$. Certainly $\alpha_0 = id_{\mathcal{O}_X}$. One has $\widetilde{\mathcal{T}}_D = \widetilde{\mathcal{T}}^{\vee} + \widetilde{\mathcal{T}}_{\Omega^{1/2}} := \{(a,b) \in \widetilde{\mathcal{T}}^{\vee} \times \widetilde{\mathcal{T}}_{\Omega^{1/2}} : \sigma(a) = \sigma(b)\}/\{\text{relations}(a,b) = (a+f, b-f) \text{ for } f \in \mathcal{O}_X\}$. Define $\alpha_1 : \widetilde{\mathrm{gr}}_1 A = A_1 = \widetilde{\mathcal{T}}^{\vee} \underset{\sim}{\rightarrow} \widetilde{\mathrm{gr}}_1 D = \widetilde{\mathcal{T}}_D$ by formula $\alpha_1(a) = (a, \sigma(a)^-)$, where $\sigma(a)^-$ is a unique element of $\widetilde{\mathcal{T}}^-_{\Omega^{1/2}}$ with $\sigma(\sigma(a)^-) = \sigma(a)$. Note that for $f \in \mathcal{O}_X$ one has $\alpha_1(fa) = f\alpha_1(a)\frac{1}{2}\sigma(a)(f)$. For arbitrary n we define $\alpha_n : \widetilde{\mathrm{gr}}_N A = S^n A_1/S^{n-2}A_1 \rightarrow \widetilde{\mathrm{gr}}_N D$ by formula $\alpha_n(a_1 \cdot \ \cdots \ \cdot a_n) = \left(\frac{1}{n!}\sum_{S \in \Sigma_n} \alpha_1(A_{S(1)} \cdot \alpha_1(a_{S(2)}) \cdot \ \cdots \ \cdot \alpha_1(a_{S(n)})\right) \mod D_{n-2}$. Here in right bracket $\cdot$ means product of differential operators. To see that this formula is correct it suffices to verify that for $f \in \mathcal{O}_X$ one has $\alpha_n(fa_1 \cdot a_2 \cdot \ \cdots \ \cdot a_n) = \alpha_n(a_1 \cdot fa_2 \cdot \ \cdots \ \cdot a_n)$ (since the formula is obviously symmetric). One has

$$\alpha_n(fa_1\cdot a_2\cdot\ \cdots\ \cdot a_n)=\frac{1}{n!}\sum_{1\le i\le n}\sum_{\substack{S\in\Sigma_n\\ S(i)=1}}\alpha_1(a_{S(1)})\cdots$$
$$\left[f\alpha_1(a_1)+\frac{1}{2}\sigma(a_1)(f)\right]\cdots\alpha_1(a_{S(n)})$$
$$=\Bigg[f\alpha_n(a_1\cdot\ \cdots\ \cdot a_n)\frac{1}{n!}\sum_{\substack{1\le i\le n\\ 1\le j<1}}\sum_{\substack{\sigma\in\Sigma_n\\ s(i)=1}}\sigma(a_{S(j)})(f)$$
$$\cdot\,\alpha_1(\widehat{a_{S(1)}})\cdots\alpha_1(\widehat{a_{S(j)}})\cdots\alpha_1(a_{S(n)})$$
$$+\frac{1}{2}\sigma(a_1)(f)a_{n-1}(a_2\cdot\ \cdots\ \cdot a_n)\Bigg]\mod D_{n-2}$$
$$=\left[f\alpha_n(a_1\cdots a_n)+\frac{1}{2}\sum_i\sigma(a_i)(f)a_{n-1}(a_1\cdots\hat{a}_i\cdots a_n)\right]$$
$$\mod D_{n-2}$$

This implies correctness; since the diagram

$$\begin{array}{ccccccccc}
0 & \to & S^{n-1}\mathcal{T}_X & \to & \tilde{\mathrm{gr}}_n D & \to & S^n\mathcal{T}_X & \to & 0\\
 & & || & & \downarrow\alpha_n & & || & & \\
0 & \to & S^{n-1}\mathcal{T}_X & \to & \tilde{\mathrm{gr}}_n A & \to & S^n\mathcal{T}_X & \to & 0
\end{array}$$

obviously commutes, our α_n is isomorphism. Put $\tilde{\sigma}. = \alpha.^{-1}$.

Note that for $D = D_{\Omega^{1/2}}$ one has $A = A_{\Omega^{1/2}} = \oplus S^i\mathcal{T}_X$. The above $\tilde{\sigma}$ obviously coincides in this case with the isomorphism from A2.4, hence it commutes with brackets. Since any tdo locally (in algebraic situation, actually, on formal neighborhood of points) is isomorphic to $D_{\Omega^{1/2}}$ and our $\tilde{\sigma}$ is natural, we see that $\tilde{\sigma}$ commutes with brackets for arbitrary D. $\square$

A2.6 Corollary. *A boundary $\delta_D : H^i(S, S^i\mathcal{T}_X) \to H^{i+1}(X, S^{j-1}\mathcal{T}_x)$ for the* *short exact sequence* $0 \to S^{j-1}\mathcal{T}_X) \to 0$ *coincides with convolution with class* $c_1(D) - \frac{1}{2}c_1(\Omega) \in H^1(X, \Omega^1_X)$. □

A.3 Descent for tdo's

Let $\pi : X \to Y$ be a morphism of smooth varieties. The corresponding morphism $\Omega^{\cdot}_Y \to \Omega^{\cdot}_X$ defines a functor $\pi^* : \Omega^{\geq 1}_Y\text{-tors} \to \Omega^{\geq 1}_Y\text{-tors}$, hence, by A1.4, A1.6, A1.10 the functors $\pi^* : \mathcal{TDO}(Y) \to \mathcal{TDO}(Y)$, $\mathcal{TCB}(Y) \to \mathcal{TCB}(X)$.

Assume π is smooth and surjective. We would like to understand how to go backwards from tdo's on X to ones on Y, i.e., how to make a descent for tdo's.

Let $(\mathcal{F}, \mathrm{curv})$ be an $\Omega^{\geq 1}_X$-torsor. It defines by push-out the "fiberwise" $\Omega^{\geq 1}_{X/Y}$-torsor $(\mathcal{F}/Y, \mathrm{curv}/Y)$, so $\mathcal{F}/Y = \mathcal{F} \bmod \pi^*\Omega^1_Y$. If $D_{\mathcal{F}}$ is a tdo on X that corresponds to $\mathcal{F}$, then sections on $\mathcal{F}/Y$ are vertical connections on $D_{\mathcal{F}}$; a vertical connection α is called integrable if $\mathrm{curv}/Y(\alpha) \in \Omega^2_{X/Y}$ vanishes.

For a section α of $\mathcal{F}/Y$ such that $\mathrm{curv}/Y(\alpha) = 0$ consider the sheaf $\mathcal{F}^\alpha := \{\gamma \in \mathcal{F} : \gamma \bmod \pi^*\Omega^1_Y = \alpha \text{ and } \mathrm{curv}(\gamma)\pi^*\Omega^2_Y \subset \Omega^2_X\}$. We will say that α is good if $\mathcal{F}^\alpha$ is nonempty: in this case $\mathcal{F}^\alpha$ is a $\pi^{-1}\Omega^1_Y$-torsor (here $\pi^{-1}\Omega^1_Y \subset \Omega^1_X$ is sheaf-theoretic inverse image of Ω^1_Y), and $\mathrm{curv}(\mathcal{F}^\alpha) \subset \pi^*\Omega^2_Y \subset \Omega^2_X$. It is easy to find an obstruction for α to be good; it lies in $H^0(Y, \Omega^1_Y \otimes \mathcal{H}^1_{DR}(X/Y))$. In particular, if fiberwise first the de Rham cohomology $\mathcal{H}^1(X/Y)$ vanishes, then α is good.

A3.1 Definition. *We will call a good section* α *a* π*-descent data for* $(\mathcal{F}, \mathrm{curv})$*, (or for a corresponding tdo, a twisted cotangent bundle...).* □

An $\Omega_X^{\geq 1}$-torsor equipped with a π-descent data form a category $\Omega_X^{\geq 1}$-tors π_i in an obvious manner; one has a similar category $\mathcal{TDO}(X)^\pi$ for tdo's. If $(\mathcal{F}_Y, \mathrm{curv}_Y)$ is an $\Omega_Y^{\geq 1}$-torsor, then the $\Omega_X^{\geq 1}$-torsor $\pi^*(\mathcal{F}, \mathrm{curv}_Y)$ carries an obvious descent data α with $(\pi^*\mathcal{F}_Y)^\alpha = \pi^{-1}\mathcal{F}$. This defines a functor $\pi^* : \Omega_Y^{\geq 1}\text{-tors} \to \Omega_X^{\geq 1}\text{-tors}^\pi$.

A3.2 Lemma. *If the fibers of π are connected, then $\pi^* : \Omega_Y^{\geq 1}\text{-tors} \to \Omega_X^{\geq 1}$-tors$^\pi$ is equivalence of categories.*

Proof: The inverse functor π_* is given by formula $\pi_*(\mathcal{F}_X, \mathrm{curv}_X; \alpha) = \pi.(\mathcal{F}_X^\alpha)$. $\square$

Certainly, we may replace in A3.2 the torsors by tdo's or twisted cotangent bundles.

A3.3 Example: Let D_Y be a tdo on Y and $\pi : X \to Y, \omega$ be the twisted cotangent bundle that corresponds to D_Y. Then $\pi^* D_Y$ carries a canonical connection ∇ with curvature ω, i.e., $\pi^* D_Y = D_{\omega_X}$ (see A1.7, A1.8). The descent data coincides with vertical part $\nabla_{X/Y}$ of ∇, hence $D_Y = \pi_*(D_{\omega_X}, \nabla_{X/Y})$. $\square$

A4 Symmetries

Let $\mathfrak{g}$ be a Lie algebra action on a smooth variety X, so we have a Lie algebra map $\nu : \mathfrak{g} \to \mathcal{T}_X$, and let D be a tdo on X.

A4.1 Definition (i) *A weak ν-action of $\mathfrak{g}$ on D is a Lie algebra map ν_D : $\mathfrak{g}\mathrm{Der}(D)$ such that for $f \in \mathcal{O}_X \subset D$, $a \in \mathfrak{g}$ one has $\nu(D)(\alpha)(f) = \nu(\alpha)(f) \in \mathcal{O}_X \subset D$.*

(ii) *A strong ν-action of $\mathfrak{g}$ on D is a Lie algebra map $\tilde{\nu}_D : \mathfrak{g} \to \widetilde{\mathcal{T}}_D$ such that $\sigma\tilde{\nu}_\phi = \nu$.* $\square$

Any strong ν-action $\tilde{\nu}_D$ defines a weak one $\nu_D := \mathrm{ad}_{\tilde{\nu}_D}$. We will say that $\tilde{\nu}_D$ lifts ν_D.

A4.2 Lemma. *Let ν_D be a weak ν-action. If either $H^1_{DR}(X) = 0$ or $H^2(\mathfrak{g}, \mathbb{C}) = 0$, then there exists a strong ν-action $\tilde{\nu}_D$ that lifts ν_D. If $H^1(\mathfrak{g}, \mathbb{C}) = 0$ (i.e., if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$) then such ν_D is unique.*

Proof: Clear. $\square$

A4.3 Examples: (i) *Let $\mathcal{L}$ be an invertible sheaf on X. A strong ν-action of $\mathfrak{g}$ on $D_{\mathcal{L}}$ is the same as a $\mathfrak{g}$-action $\tilde{\nu}_D$ of $\mathfrak{g}$ on $\mathcal{L}$ that lifts ν.*

(ii) *Let ω be a closed 2-form on X, and D_ω be the tdo with connection ∇_ω such that* $\mathrm{curv}\nabla_\omega = \omega$ *(see A1.7). Let $\tilde{\nu}_\omega : \mathfrak{g} \to \widetilde{\mathcal{T}}_\omega = \widetilde{\mathcal{T}}_{D_\omega}$ be a strong ν-action, so for $\alpha \in \mathfrak{g}$ one has $\tilde{\nu}_\omega(\alpha) = \nabla_\omega \nu(\alpha) + \varphi(\alpha)$, where $\varphi(\alpha) \in \mathcal{O}_X$. This action preserves ∇_ω (which means that $[\tilde{\nu}_\omega(\alpha), \nabla(\tau)] = \nabla([\nu(\alpha), \tau])$ for $\alpha \in \mathfrak{g}$, $\tau \in \mathcal{T}_X$) precisely if $\varphi(\alpha)$ is an ω-Hamiltonian for $\nu(\alpha)$, i.e., if $d\varphi(a) = \nu(\alpha) - \omega$. We will call such $\tilde{\nu}_\omega$ (or a pair $(\nu, \varphi) : \mathfrak{g} \to \mathcal{T}_X \times \mathcal{O}_X$) an ω-Hamiltonian action of $\mathfrak{g}$, or ω-Hamiltonian lifting of ν.* $\square$

A4.4. Assume we have a weak ν-action ν_D, and M is a D-module. A ν_D-action of $\mathfrak{g}$ on M is a Lie algebra map $\nu_M : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} M$ such that for $\partial \in D$, $\alpha \in \mathfrak{g}$, $m \in M$ one has $\nu_M(\alpha)\partial - \partial\nu_M(\alpha))m = \nu_D(\alpha)(\partial)m$.

Assume now that we have a strong lifting $\tilde{\nu}_D : \mathfrak{g} \to \widetilde{\mathcal{T}}_D$ of ν_D. Then one has a canonical ν_D-action ν^0_M on any D-module M defined by formula $\nu^0_M(\alpha)m = \tilde{\nu}_D(\alpha)m$. More generally, for aany ν_D-action ν_M of $\mathfrak{g}$ on M

consider the map $[\nu_M] : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}} M$, $[\nu_m](\alpha) := \nu_M(\alpha) - \nu^0_M(\alpha)$.

A4.5 Lemma. *The operators $[\nu_M](\alpha)$ commute with D-action and $[\nu_M] : \mathfrak{g} \to \mathrm{End}_D M$ is a Lie algebra map, i.e., $[\nu_M]$ is an action of $\mathfrak{g}$ on D-module M. The map $\nu_M \mapsto [\nu_M]$ is a 1–1 correspondence between the set of ν_D-actions of $\mathfrak{g}$ on M and the one of actions of $\mathfrak{g}$ on M as on D-module.*

Proof: Clear. $\square$

Appendix B

Chern Classes

In this Appendix we recall an explicit Weil algebra construction of Chern classes for de Rham and Deligne-type cohomology. Below "variety" means either a smooth algebraic or analytic variety. Starting from B4 we assume that we are in an analytic situation.

B1 Weil Algebra

We will start with some notations.

B1.1. For a variety X denoted by $\mathcal{P}(X)$ a category whose objects are Ω^1_X-extensions. These are short exact sequences $P = (0 \to \Omega^1_X(P) \to \widetilde{\Omega}^1 \to M(P) \to 0)$ of coherent locally free $\mathcal{O}_X$-modules; morphsms are obvious. The categories $\mathcal{P}(X)$ form a fibered category over category of varieties: for a morphism $\pi : X \to Y$ of varieties we have a pullback functor $\pi^* : \mathcal{P}(Y) \to \mathcal{P}(X)$. Namely, for $P = (0 \to \Omega^1_Y \to \widetilde{\Omega}^1(P) \to M(P) \to 0)$ one has $\pi^*(P) = (0 \to \Omega^1_X \to \widetilde{\Omega}^1(\pi^*P) \to \pi^*M(P) \to 0)$, where $\widetilde{\Omega}^1(\pi^*P)$ comese from co-Cartesian square

$$\begin{array}{ccc}
\pi^*\Omega^1_Y & \stackrel{d\pi}{\to} & \Omega^1_X \\
\downarrow & & \downarrow \\
\pi^*\widetilde{\Omega}^1(P) & \to & \widetilde{\Omega}^1(\pi^*P).
\end{array}$$

B1.2. For $P \in \mathcal{P}(X)$ let $\widetilde{\Omega}(P)$ be a sheaf of commutative differential graded (cdg for short) algebras generated by a subalgebra $\mathcal{O}_X$ in degree 0 and an

$\mathcal{O}_X$-module $\widetilde{\Omega}^1(P)$ in degree 1 subject to only relation: for $F \in \mathcal{O}_X$ its differential coincides with usual differential $df \in \Omega^1_X \subset \Omega^1(P)$. Denote by F^1 the *dg*-ideal $\widetilde{\Omega}^{\geq 1}(P) \subset \widetilde{\Omega}^{\cdot}(P)$; its powers form a filtration F^i on $\widetilde{\Omega}^{\cdot}(P)$. The filtered cdg algebra $\widetilde{\Omega}^{\cdot}(P)$ depends on P in a functorial way.

B1.3 Examples: (i) Let P_0 be a trivial Ω^1_X extension, $\widetilde{\Omega}^1(P_0) = \Omega^1_X$. One has $\widetilde{\Omega}^{\cdot}(P_0) = \Omega^{\cdot}_X$, $F^i\widetilde{\Omega}^{\cdot}(P_0) = \Omega^{\geq i}_X$. Since P_0 is a universal object in $\mathcal{P}(X)$ we see that $\widetilde{\Omega}^{\cdot}(P)$'s are $\Omega^{\cdot}_X$-algebras.

(ii) If X is a point, then $\Omega^1_X = 0$ and $P \in \mathcal{T}(X)$ reduces to a vector space $M = M(P)$. The algebra $\widetilde{\Omega}^{\cdot}(M)$ is a commutative graded algebra freely generated by two copies of M: $M^{(1)}$ in degree one and $M^{(2)}$ in degree two. The differential is determined by rule: for $m \in M^{(1)}$ one has $dm = m \in M^{(2)}$. Hence $\widetilde{\Omega}^i(M) = \oplus_{a+2b=i}\Lambda^q M \otimes S^b M$, $d(m_1 \wedge \cdots \wedge m_a \otimes m'_1 \cdot \cdots \cdot m'_b) = \sum(-1)^i m_1 \wedge \cdots \wedge \widehat{m}_i \wedge \cdots \wedge m_a \otimes m_i \cdot, m'_1 \cdot \cdots \cdot m'_b$. $\square$

B1.4 Lemma. (i) *For a morphism $\pi : X \to Y$ and $P \in \mathcal{P}(Y)$ one has $\widetilde{\Omega}^{\cdot}(\pi^*P) = \pi^*\widetilde{\Omega}^{\cdot}(P) := \Omega^{\cdot}_X \otimes_{\pi^{-1}\Omega^{\cdot}_Y} \pi^{-1}\widetilde{\Omega}^{\cdot}(P)$, where π^{-1} is sheaf-theoretic inverse image.*

(ii) *For $P \in \mathcal{T}(X)$ the complex $F^1/F^2 = F^1\widetilde{\Omega}^{\cdot}(P)/F^2\widetilde{\Omega}^{\cdot}(P)$, coincides with complex $\widetilde{\Omega}^1(P) \to M(P)$ supported in degrees 1,2.*

(iii) *A natural morphism $S^*(F^1/F^2) \to gr^*_F\widetilde{\Omega}^{\cdot}(P)$ is isomorphism. Here $S^*(F^1/F^2)$ is a free commutative graded* dg *algebra generated by F^1/F^2. Note that, according to (ii), $S^i(F^1/F^2)$ is the complex $\Lambda^i(\widetilde{\Omega}^1(P)) \to \Lambda^{i-1}(\widetilde{\Omega}^1(P)) \otimes M(P) \to \cdots \to \widetilde{\Omega}^1(P) \otimes S^{i-1}M(P) \to S^iM(P)$ supported in degrees $i, \ldots, 2i$.*

(iv) *A canonical morphism $\Omega^{\cdot}_X \to \widetilde{\Omega}^{\cdot}(P)$ is a filtered quasi-isomorphism.*

Proof: (i) follows from definition, (ii), (iii) follows from (i) and B1.3(ii) since locally any P comes from a point, (iv) follows from (iii) since the sequence $0 \to \Omega^i_X \to \Lambda^i\widetilde{\Omega}^1(P) \to \cdots \to S^iM(P) \to 0$ is exact. $\square$

B1.5. Let G be an algebraic group, $\mathfrak{g}$ =Lie G, and $p : \mathcal{E} \to X$ be a G-torsor on our variety X. Consider the sheaf $\widetilde{\Omega}^1_{X,\mathcal{E}} = (p_*\Omega^1_{\mathcal{E}})^G$ of G-invariant 1-forms. This is an $\mathcal{O}_X$-module; we have a short exact sequence $P_{\mathcal{E}} = (0 \to \Omega^1_X \overset{dp}{\to} \widetilde{\Omega}^1_{X,\mathcal{E}} \to \mathfrak{g}^*_{\mathcal{E}} \to 0)$, where $\mathfrak{g}^*_{\mathcal{E}} = (p_*\Omega^1_{\mathcal{E}/X})^G$ is $\mathcal{E}$-twist of $\mathfrak{g}^* \otimes \mathcal{O}_X$ with respect to coadjoint action of G. Put $\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}} = \widetilde{\Omega}^{\cdot}(P_{\mathcal{E}})$. This is a filtered commutative differential graded $\Omega^{\cdot}_X$-algebra such that a canonical map $\Omega^{\cdot}_X \to \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ is a filtered quasi-isomorphism.

B1.6 Definition. *$\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ is called Weil algebra of .* $\square$

B1.7 Lemma. *$\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ depends on $\mathcal{E}$ in a functorial way. If $\pi : X \to Y$ is a morphism of varieties, $\mathcal{E}_Y$ is a G-torsor on Y, and $\mathcal{E}_X = \pi^*\mathcal{E}_Y$, then $\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}_X} = \pi^*\widetilde{\Omega}^{\cdot}_{Y,\mathcal{E}_Y}$.*

Proof: Follows from B1.4(i) since $P_{\mathcal{E}_Y} = \pi^*P_{\mathcal{E}_X}$. $\square$

The Weil algebra carries a canonical bigrading. To define it consider the *cdg* algebra $(p_*\Omega^{\cdot}_{\mathcal{E}})^G$ of all G-invariant differential forms. Clearly $(p_*\Omega^i_{\mathcal{E}})^G = \Lambda^i\widetilde{\Omega}^1_{X,\mathcal{E}}$. Denote by d' the differential on $\Lambda^{\cdot}\widetilde{\Omega}^1_{X,\mathcal{E}}$ that comes from this isomorphism. For $\nu \in \widetilde{\Omega}^1_{X,\mathcal{E}}$ put $d''(\nu) = d(\nu) - d'(\nu) \in \widetilde{\Omega}^2_{X,\mathcal{E}}$; here $d'(\nu) \in \Lambda^2\widetilde{\Omega}^1_{X,\mathcal{E}} = F^2\widetilde{\Omega}^2_{X,\mathcal{E}} \subset \widetilde{\Omega}^2_{X,\mathcal{E}}$. Clearly $d''\nu = 0$ for $\nu \in \Omega^1_X$, and the isomorphism $\widetilde{\Omega}^2_{X,\mathcal{E}}/F^2\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}} \simeq \mathfrak{g}^*_{\mathcal{E}} = \widetilde{\Omega}^1_{X,\mathcal{E}}/\Omega^1_X$ (see B1.4(ii)) identifies $d''\nu$ mod F^2 with ν mod Ω^1_X. Hence d'' defines a canonical $\mathcal{O}_X$-linear embedding $\alpha : \mathfrak{g}^*_{\mathcal{E}} \hookrightarrow \widetilde{\Omega}^2_{X,\mathcal{E}}$, $d''(\nu) = \alpha(\nu \mod \Omega^1_X)$ such that $\widetilde{\Omega}^2_{X,\mathcal{E}} = \Lambda^2\widetilde{\Omega}^1_{X,\mathcal{E}} \oplus \alpha(\mathfrak{g}^*_{\mathcal{E}})$. Let $\Lambda^{\cdot}\widetilde{\Omega}^1_{X,\mathcal{E}} \otimes S^*\mathfrak{g}^*_{\mathcal{E}}$ be a free commutative graded algebra with generators $\widetilde{\Omega}^1_{X,\mathcal{E}}$ in degree 1 and

$\mathfrak{g}^*_{\mathcal{E}}$ in degree 2, and $\tilde{\alpha} : \Lambda^{\cdot}\widetilde{\Omega}^1_{X,\mathcal{E}} \otimes S^*\mathfrak{g}^*_{\mathcal{E}} \to \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ be a morphism of commutative graded algebras which is equal to $id_{\widetilde{\Omega}^1_{X,\mathcal{E}}}$ on $\widetilde{\Omega}^1_{X,\mathcal{E}}$ and to α on $\mathfrak{g}^*_{\mathcal{E}}$.

B1.8 Lemma. *This $\tilde{\alpha}$ is isomorphism.*

Proof: Consider filtration F^i on $\Lambda^{\cdot} \otimes S^*$ by powers of augmentation ideal. By B1.4 (iii) $\tilde{\alpha}$ induces isomorphism between gr_F's. $\square$

Put $\widetilde{\Omega}^{a,b}_{X,\mathcal{E}} := \tilde{\alpha}(\Lambda^{a-b}2\widetilde{\Omega}^1_{X,\mathcal{E}} \otimes S^b\mathfrak{g}^*_{\mathcal{E}}) \subset \widetilde{\Omega}^{a+b}_{X,\mathcal{E}}$.

B1.9 Lemma. *This is a canonical bigrading on $\widetilde{\Omega}^{\cdot}_{Y,\mathcal{E}}$ compatible with filtration $F^{\cdot}$. In other words, one has $\widetilde{\Omega}^n_{X,\mathcal{E}} = \oplus_{a+b=n}\widetilde{\Omega}^{a,b}_{X,\mathcal{E}}$, $F^i\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}} = \oplus_{a\geq i}\widetilde{\Omega}^{a,b}_{X,\mathcal{E}}$, $d = d'' + d'' : \widetilde{\Omega}^{a,b}_{X,\mathcal{E}} \to \widetilde{\Omega}^{a+1,b}_{X,\mathcal{E}} + \widetilde{\Omega}^{a,b+1}_{X,\mathcal{E}}$.*

Proof: Clear. $\square$

B1.10 Example: Assume that X is a point, so $\mathcal{E}$ is trivial. One has $\widetilde{\Omega}^{a,b}_{X,\mathcal{E}} = \Lambda^{a-b}\mathfrak{g}^* \oplus S^b\mathfrak{g}^*$. The differential $d' = \Lambda^a\mathfrak{g}^* \otimes S^b\mathfrak{g}^* \to \Lambda^{a+1}\mathfrak{g}^* \otimes S^b\mathfrak{g}^*$ is the differential in the cochain complex of $\mathfrak{g}$ with values in symmetric power of coadjoint representation. The differential $d'' : \Lambda^a\mathfrak{g}^* \otimes S^b\mathfrak{g}^* \to \Lambda^{a-1}\mathfrak{g}^* \otimes S^{b+1}\mathfrak{g}^*$ is Koszul differential. We see that $\widetilde{\Omega}$ is a classical Weil algebra (see, e.g., [ ]). $\square$

Since and Ad_G-invariant polynomial on $\mathfrak{g}$ defines a polynomial on any ad-twisted form of $\mathfrak{g}$, we have a canonical map $w^i : S^i(\mathfrak{g}^*)^G \to S^i(\mathfrak{g}^*_{\mathcal{E}})^G = \widetilde{\Omega}^{i,i}_{X,\mathcal{E}} \subset F^i\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ called Weil homomorphism.

B1.11 Lemma. *The image of w consists of cycles, i.e., $w^{\cdot} : S^{\cdot}(\mathfrak{g}^*[-2])^G = \oplus_i S^i(\mathfrak{g}^*)^G[-2i] \to \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ is a morphism of cdg algebras.*

Proof: The fact is local, hence we may assume that $\mathcal{E}$ is trivial, i.e., $\mathcal{E}$ is a pullback of a G-torsor $\mathcal{E}'$ on a point. By functoriality it suffices to prove B1.11 for $\mathcal{E}'$, which follows from B1.10.

B2 De Rham Chern Classes

Let $\mathcal{E}$ be a G-torsor on X. By B1.4 (iv) one has a canonical isomorphism $H^{\cdot}(X, F^i\widetilde{\Omega}^{\cdot}_X) \underset{\sim}{\rightarrow} H^{\cdot}(X, F^i\Omega^{\cdot}_{X,\mathcal{E}})$. By B1.11 one has a canonical ring homomorphism $w^i : S^i(\mathfrak{g}^*)^G \rightarrow H^{2i}(X, F^i\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}})$. Let $\omega_{\mathcal{E}}$ be the composition $S^i(\mathfrak{g}^*)^G \rightarrow H^{2i}(X, F^i\Omega^{\cdot}_X)$. This is Weil homomorphism in de Rham cohomology.

Let us consider the universal situation. Let $BG.$ be simplicial classifying space of G, and $p : \mathcal{E}_{un} = \Delta G. \rightarrow BG.$ be universal torsor. So one has $\Delta G_n = G^{n+1}$, BG_n is a quotient of ΔG_n modulo diagonal action of G, and the simplicial arrows are the obvious ones. The Chern character of $\mathcal{E}_{un}$ defines the ring homomorphism

$$w^i_{\mathcal{E}_{un}} : S^i(\mathfrak{g}^*)^G \rightarrow H^{2i}(BG, F^i\Omega^{\cdot}_{BG}) \rightarrow H^i(BG., \Omega^i_{BG}).$$

B2.1 Lemma. *Assume that G is reductive. Then the map $w^i_{\mathcal{E}_{un}}$ is isomorphism and $H^j(BG, \Omega^i_{BG.}) = 0$ for $j \neq i$.*

Proof: Consider first the algebraic situation. One has the exact sequence $0 \rightarrow \Omega^i_{BG.} \rightarrow \Delta^i\widetilde{\Omega}^1(P_{\mathcal{E}_{un}}) \rightarrow \Lambda^{i-1}\widetilde{\Omega}^1(P_{\mathcal{E}_{un}}) \otimes \mathfrak{g}^*_{\mathcal{E}_{un}} \rightarrow \ldots \rightarrow S^i\mathfrak{g}^*_{\mathcal{E}_{un}} \rightarrow 0$ (which is i-th symmetric power of the short acyclic complex $0 \rightarrow \Omega^1_{BG.} \rightarrow \widetilde{\Omega}^1(P_{\mathcal{E}_{un}}) \rightarrow \mathfrak{g}^*_{\mathcal{E}_{un}} \rightarrow 0$, see B1.4(iii)). Note that BG_n is affine and

$$H^0(BG_n, \Lambda^a\widetilde{\mathcal{T}}^1(P_{\mathcal{E}_{un}}) \otimes S^b fg^*_{\mathcal{E}_{un}}) = [H^0(\Delta G_n, \Omega^a_{\Delta G_n}) \otimes S^b\mathfrak{g}^*]^G.$$

Since $\Delta G.$ is "contractible simplex with set of vertices G," one has $H^i(\Delta G., \Omega^a_{\Delta G.}) = 0$ unless $i = 0$, $a = 0$, and $H^0(\Delta G., \mathcal{O}_{\Delta G.}) = \mathbb{C}$. Since our group is reductive, this implies $H^i(BG., \Lambda^a \widetilde{\Omega}^1(P_{\mathcal{E}_{un}}) \otimes S^b \mathfrak{g}^*_{\mathcal{E}_{un}}) = 0$ unless $i = 0$, $a = 0$, and $H^0(BG, S^b \mathfrak{g}^*_{\mathcal{E}_{un}}) = [S^b \mathfrak{g}^*]^G$.

The above exact sequence shows that

$$H^{\cdot}(BG., \Omega^i_{BG.}) = H^{\cdot - i}(BG., S^i \mathfrak{g}^*_{\mathcal{E}_{un}}),$$

and the lemma is proven. In analytic situation one should use the averaging along a maximal compact subgroup of G to see that acyclicity of the complex $H^0(\Delta G., \Omega^i_{\Delta G.})$ implies the acyclicity of complex of G-invariants. $\square$

B2.2 Corollary. *The maps*

$$S^i(\mathfrak{g}^*)^G \xrightarrow{w_{\mathcal{E}_{un}}} H^{2i}(BG., F^i \Omega^{\cdot}_{BG.}) \to H^{2i}(BG., \Omega^{\cdot}_{BG.}) = H^{2i}_{DR}(BG.)$$

are isomorphisms. The odd-dimensional de Rham cohomology of $BG.$ vanishes. The map $H^j(BG., F^i \Omega^{\cdot}_{BG.}) \to H^j_{DR}(BG.)$ is isomorphism for $j \geq 2i$ for $j < 2i$, $H^j(BG., F^i \Omega^{\cdot}_{BG.}) = 0$. $\square$

B3 Connections

Let $\mathcal{E}$ be a G-torsor on a variety X. A connection ∇ on $\mathcal{E}$ is an $\mathcal{O}_X$-linear splitting of $P_{\mathcal{E}}$ (see B1.5), i.e., ∇ is an $\mathcal{O}$-linear map $\widetilde{\Omega}^1_{X,\mathcal{E}} \to \Omega^1_X$ such that $\nabla(df) = df \in \Omega^1_X$ for $f \in \mathcal{O}_X$. One may consider ∇ as a morphism $P_{\mathcal{E}} \to P_0$ (see B1.3(i)), hence it extends to a morphism of dg algebras $\widetilde{\nabla} : \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}} \to \Omega^{\cdot}_X$ left inverse to a canonical embedding $\widetilde{\Omega}^{\cdot}_X \hookrightarrow \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$. The morphism $\widetilde{\nabla}^{11} = \widetilde{\nabla}|_{\widetilde{\Omega}^{1,1}_{X,\mathcal{E}}} : \widetilde{\Omega}^{1,1}_{X,\mathcal{E}} = \mathfrak{g}^*_{\mathcal{E}} \to \Omega^2_X$, $\widetilde{\nabla}^{1,1} \in \mathfrak{g}_{\mathcal{E}} \otimes \Omega^2_X$, is curvature form of our connection. We see that $\widetilde{\widetilde{\nabla}} \circ w$ sends an invariant polynomial $\varphi \in S^i(\mathfrak{g}^*)^G$ to $\varphi(\widetilde{\widetilde{\nabla}}^{11}) \in \Omega_X^{2ic\ell}$.

B4 "Universal" Chern Classes

In this section we give a universal construction that matches integral topological Chern classes with de Rham ones. From now on we assume that our varieties are analytic ones (so we will consider classical topology, not a Zariski one). Our group G is reductive.

B4.1 Let X be a variety, and $\mathcal{E}$ be a G-torsor on X. Consider the embeddings of constant sheaves

$$\mathbb{Z}(i) \hookrightarrow \widetilde{\Omega}^\bullet_{X,\mathcal{E}} \overset{w^i_{\mathcal{E}}}{\hookleftarrow} S^i(\mathfrak{g}^*)^G[-2i];$$

here $\mathbb{Z}(i) := (2\pi\sqrt{-1})^i\mathbb{Z} \subset \mathbb{C} \subset \mathcal{O}_X$. Put

$$U_{\mathcal{E}}(i) := \mathrm{Cone}(\mathbb{Z}(i) \oplus S^i(\mathfrak{g}^*)^G[-2i] \xrightarrow{(+,-)} \widetilde{\Omega}^\bullet_{X,\mathcal{E}})[-1];$$

the arrow is difference of the embeddings. One has canonical triangles in derived category of sheaves on X (recall that one has canonical quasi-isomorphisms

$$\mathbb{C} \underset{\sim}{\to} \Omega^\bullet_X \underset{\sim}{\to} \widetilde{\Omega}^\bullet_{X,\mathcal{E}}\mathbb{C}/\mathbb{Z}(i) \underset{\sim}{\xrightarrow{\exp}} \mathbb{C}^*(i-1) = \mathbb{C}^* \otimes \mathbb{Z}(i-1)).$$

B4.2.

$$\cdots \to \mathbb{C}[-1] \to U_{\mathcal{E}}(i) \xrightarrow{(\epsilon_{\mathbb{Z}},\epsilon_p)} \mathbb{Z}(i) \oplus S^i(\mathfrak{g}^*)^G[-2i] \to \cdots$$

$$\cdots \to \mathbb{C}^*[-1] \to U_{\mathcal{E}}(i) \xrightarrow{(\epsilon_{\mathbb{Z}},\epsilon_p)} S^i(\mathfrak{g}^*)^G[-2i] \to \cdots$$

The groups $H^{\cdot}(X,\mathcal{U}_{\mathcal{E}}(i))$ are clearly functorial with respect to $(X,\mathcal{E})$. The long exact sequences that correspond to B4.2 imply

B4.3 Lemma. (i) *A canonical morphism* $H^{j-1}(X,\mathbb{C}^*)(i-1) \to H^j(X,\mathcal{U}_{\mathcal{E}}(i))$ *is isomorphism for* $j < 2i$. *One has a short exact sequence*

$$0 \to H^{2i-1}(X,\mathbb{C}^*)(i-1) \to H^{2i}(X,\mathcal{U}_{\mathcal{E}}(i)) \to S^i(\mathfrak{g}^*)^G_{\mathbb{Z},\mathcal{E}} \to 0,$$

where $S^i(\mathfrak{g}^*)^G_{\mathbb{Z},\mathcal{E}} subset S^i(\mathfrak{g}^*)^G$ *consists of those polynomials* φ *that* $\int_\gamma ch^i(\mathcal{E})(\varphi) \in \mathbb{Z}(i) = (2\pi\sqrt{-1})^i\mathbb{Z} \subset \mathbb{C}$ *for any* $\gamma \in H_{2i}(X,\mathbb{Z})$.

(ii)*If* $\pi : X \to Y$ *is a morphism of varieties (or simplicial varieties) such that* $\pi^* : H^{\cdot}(Y,\mathbb{Z}) \to H^{\cdot}(X,\mathbb{Z})$ *is an isomorphism, then for any* G*-torsor* $\mathcal{E}_Y$ *on* Y, $\mathcal{E}_X := \pi^*\mathcal{E}_Y$, *a canonical map* $\pi^* : H^{\cdot}(Y,\mathcal{U}_{\mathcal{E}_Y}(i)) \to H^{\cdot}(X,\mathcal{U}_{\mathcal{E}_X}(i))$ *is isomorphism.* □

B4.4 Remark: The same formulas that define product in Deligne cohomology (see [B], [EV]) define a canonical homotopy associative and commutative product $\mathcal{U}_{\mathcal{E}}(i)\otimes\mathcal{U}_{\mathcal{E}}(j) \to \mathcal{U}_{\mathcal{E}}(i+j)$ such that the projection $\epsilon_{\mathbb{Z}} : \mathcal{U}_{\mathcal{E}}(\cdot) \to \mathbb{Z}(\cdot)$, $\epsilon_{\mathfrak{g}} : \mathcal{U}_{\mathcal{E}}(\cdot) \to S^{\cdot}(\mathfrak{g}^*)^G[2\cdot]$ commute with multiplication. □

Consider a universal G-torsor $\mathcal{E}_{un}$ on BG.

B4.5 Lemma. *A canonical morphism* $\epsilon_{\mathbb{Z}} : H^{2i}(BG.,\mathcal{U}_{\mathcal{E}_{un}}(i)) \to H^{2i}(BG,\mathbb{Z}(i))$ *is isomorphism.*

Proof: By B4.2 we have a long exact sequence $H^{2i-1}(BG.,\mathbb{C}) \to H^{2i}(BG,\mathcal{U}_{\mathcal{E}_{un}}(i)) \to H^{2i}(BG,\mathbb{Z}(i)) \oplus S^i(\mathfrak{g}^*)^G \to H^{2i}(BG,\mathbb{C})$. Since $H^{2i-1}(BG.,\mathbb{C}) = 0$ and $S^i(\mathfrak{g}^*)^G \to H^{2i}(BG,\mathbb{C})$ is isomorphism (see B2.2), we get the lemma. □

B4.6. Let us construct a "universal" Weil homomorphism. Let $\mathcal{E}$ be a G-torsor on X. Put $X^\vee_{\mathcal{E}} := G\backslash\mathcal{E} \times \mathcal{E}_{un}$, here G acts on $\mathcal{E} \times \mathcal{E}_{un}$ in a diagonal way. One has two projections $X \xleftarrow{\pi_X} X^\vee_{\mathcal{E}} \xrightarrow{\pi_{BG}} BG$. and an obvious isomorphism $\pi^*_X\mathcal{E} \simeq \pi^*_{BG.}\mathcal{E}_{un}$. Note that π_X is a fibration with "contractible" fibers isomorphic to $\mathcal{E}_{un} = DeltaG.$, hence $\pi^*_X : H^\cdot(X, \mathbb{Z}) \to H^\cdot(X^\vee, \mathcal{E}, \mathbb{Z})$ is isomorphism; by B4.3(ii) $\pi^*_X : H^\cdot(X, \mathcal{U}_{\mathcal{E}}(i)) \to H^\cdot(X^\vee, \mathcal{E}, \mathcal{U}_{\pi^*\mathcal{E}}(i))$ are also isomorphisms. Denote by

$$w_{\mathcal{E},\mathcal{U}} : H^{2i}(BG., \mathbb{Z}(i)) \to H^{2i}(X, \mathcal{U}_{\mathcal{E}}(i))$$

the composition

$$\begin{aligned} H^{2i}(BG., \mathbb{Z}(i)) \xleftarrow[\sim]{\epsilon_{\mathbb{Z}}} H^{2i}(BG., \mathcal{U}_{\mathcal{E}}(i)) \xleftarrow{\pi^*_{BG}} H^{2i}(X^\vee_{\mathcal{E}}, \mathcal{U}_{\pi^*_{BG}\mathcal{E}_{un}}(i)) \\ = H^{2i}(X^\vee_{\mathcal{E}}, \mathcal{U}_{\pi^*_X\mathcal{E}}(i)) \xleftarrow[\sim]{\pi^*_X} H^{2i}(X, \mathcal{U}_{\mathcal{E}}(i)) \end{aligned}$$

This is "universal" Weil homomorphism. Clearly $w_{\mathcal{E}\mathbb{Z}} = \epsilon_{\mathbb{Z}} \circ w_{\mathcal{E}\mathcal{U}} : H^{2i}(BG, \mathbb{Z}(i)) \to H^{2i}(X, \mathbb{Z}(i))$ coincides with usual topological characteristic class map. By B4.5 and the above construction our $w_{\mathcal{E}\mathcal{U}}$ is the only functorial "lifting" of $w_{\mathcal{E}\mathbb{Z}}$ to $\mathcal{U}$-cohomology. Also $w_{\mathcal{E}\mathcal{U}}$ is ring homomorphism (see B4.4).

The classes $w_{\mathcal{E}\mathcal{U}}$ take values in $\mathcal{U}_{\mathcal{E}}$-groups that depend on $\mathcal{E}$ themselves. We will use them to produce classes in Deligne-type cohomology.

B.5 Deligne Cohomology Chern Classes

We will use a naive version of Deligne cohomology, see [B], [EV].

B5.1 Let X be an analytic variety. The Deligne complex $\mathcal{D}(i)_X$ is Cone $(\mathbb{Z}(i) \oplus F^i\Omega^{\cdot}_X \xrightarrow{(+,-)} \Omega^{\cdot}_X)[-1]$, where the arrow is difference of an obvious embeddings; the Deligne cohomology groups are $H^j_{\mathcal{D}}(X, \mathbb{Z}(i)) := H^j(X, \mathcal{D}(i)_X)$. So we have a canonical map $\epsilon_{\mathbb{Z}} : \mathcal{D}(i)_X \to \mathbb{Z}(i)$, $\epsilon_F : \mathcal{D}(i)_X \to F^i\Omega^{\cdot}_X$ and the long exact sequences

$$\cdots \longrightarrow H^{j-1}(X, \mathbb{C}) \longrightarrow H^j_{\mathcal{D}}(X, \mathbb{Z}(i)) \xrightarrow{\epsilon_{\mathbb{Z}}+\epsilon_F} H^j(X, \mathbb{Z}(i)) \oplus H^j(X, F^i\Omega^{\cdot}_X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{j-1}(X, F^i\Omega^{\cdot}_X) \to H^{j-1}(X, \mathbb{C}^*)(i-1) \to H^j_{\mathcal{D}}(X, \mathbb{Z}(i)) \longrightarrow \cdots$$

Let ε be a G-torsor on X. The embedding

$$\Omega^{\cdot}_X \hookrightarrow \widetilde{\Omega}^{\cdot}_{X,\mathcal{E}} := \mathrm{Cone}(\mathbb{Z}(i) \oplus F^i\widetilde{\Omega}_{X,\mathcal{E}} \xrightarrow{(+,-)} \widetilde{\Omega}_{X,\mathcal{E}})[-1].$$

Since $w(S^i(\mathfrak{g}^*)^G) \subset F^i\widetilde{\Omega}^{\cdot}_{X,\mathcal{E}}$ we have a canonical embedding $\mathcal{U}_{\mathcal{E}}(i) \hookrightarrow \mathcal{D}(i)_{X,\varepsilon}$ which is identity on $\mathbb{Z}(i)$ and $\widetilde{\Omega}_{X,\mathcal{E}}$-components and coincides with w on $S^i(\mathfrak{g}^*)^G$. This embedding commutes with multiplication on $\mathcal{D}$- and $\mathcal{U}$-complexes (see B4.4, [B],[EV]). Denote by $w_{\mathcal{E}\mathcal{D}}$ the composition $H^{2i}(BG, \mathbb{Z}(i)) \xrightarrow{\omega_{\mathcal{E}\mathcal{U}}} H^{2i}(X, \mathcal{U}_{\mathcal{E}(i)}) \longrightarrow H^{2i}(X, \mathcal{D}(i)_{X\mathcal{E}}) = H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i))$. This is Weil homomorphism in naive Deligne cohomology.

B.6 Cheeger-Simons Cohomology

Let X be an analytic variety. Consider a complex $\mathcal{CS}(i)_X := \mathrm{Cone}(\mathbb{Z}(i) \oplus F^{2i}\Omega^{\cdot}_X \xrightarrow{(+,-)} \Omega^{\cdot}_X)[-1]$ (so $\mathcal{CS}(i)_X$ coincides with $\mathcal{D}(2i)_X(-i)$). We will call $\mathcal{CS}(i)_X$ a *Cheeger-Simons complex* and the corresponding groups $H^{\cdot}_{\mathcal{CS}}(X, \mathbb{Z}(i)) := H^{\cdot}(X, \mathcal{CS}(i)_X)$ *Cheeger-Simons cohomology.* We have a canonical morphism $\epsilon_{\mathbb{Z}} : \mathcal{CS}(i)_X \longrightarrow \mathbb{Z}(i)$, $\epsilon_F : \mathcal{CS}(i) \longrightarrow F^{2i}$ and a long exact sequence.

B6.1.

$$\cdots \longrightarrow H^{j-1}(X,\mathbb{C}) \longrightarrow H^j_{\mathcal{CS}}(X,\mathbb{Z}(i)) \xrightarrow{\epsilon_{\mathbb{Z}}+\epsilon_F} H^j(X,\mathbb{Z}(i))\oplus H^j(X,F^{2i}\Omega^{\cdot}_X) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{j-1}(X,F^{2i}\Omega^{\cdot}_X) \longrightarrow H^{j-1}(X,\mathbb{C}^*)(i-1) \longrightarrow H^j_{\mathcal{CS}}(X,\mathbb{Z}(i)) \xrightarrow{\epsilon_F} \cdots .$$

In particular, the map $H^{j-1}(X,\mathbb{C}^*)(i-1) \longrightarrow H^j_{\mathcal{CS}}(X,\mathbb{Z}(i))$ is isomorphism for $j < 2i$ and for $j = 2i$ one has a short exact sequence.

B6.2

$$0 \to H^{2i-1}(X,\mathbb{C}^*)(i-1) \longrightarrow H^{2i}_{\mathcal{CS}}(X,\mathbb{Z}(i)) \xrightarrow{\epsilon_F} H^0(X,\Omega^{2icl}_X)_{\mathbb{Z}(i)} \to 0$$

where $H^0(X,\Omega^{2icl}_X)_{\mathbb{Z}(i)}$ is the space of all closed holomorphic $2i$-forms ν on X such that $\int_\gamma \nu \in \mathbb{Z}(i) \subset \mathbb{C}$ for any $\gamma \in H_{2i}(X,\mathbb{Z})$.

B6.3. Let ∇ be a connection on a G-torsor $\mathcal{E}$. By B3 we have a commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}(i) & & = & \mathbb{Z}(i) & \\
 & \searrow & & & \searrow \\
 & & \widetilde{\Omega}^{\cdot}_{X,\varepsilon} & \xrightarrow{\widetilde{\nabla}} & \Omega^{\cdot}_X \\
 & \nearrow & & & \nearrow \\
S^i(\mathfrak{g}^*)^G[-2i] & & \longrightarrow & F^{2i}\Omega^{\cdot}_X &
\end{array}$$

where the lowest horizontal arrow maps an invariant polynomial φ to its value $\varphi(\widetilde{\nabla}^{11})$ on curvature form of ∇. This diagram defines a morphism $\gamma\nabla : \mathcal{U}_{\mathcal{E}}(i) \to \mathcal{C}S(i)_X$, hence the map $w_{(\mathcal{E},\nabla)\mathcal{C}S} = \gamma\nabla \circ w_{\mathcal{E}\mathcal{U}} : H^{2i}(BG.,\mathbb{Z}(i)) \to H^{2i}_{\mathcal{C}S}(X,\mathbb{Z}(i))$. This is the Cheeger-Simons class of a torsor with connection. Clearly $\epsilon_{\mathbb{Z}} \circ ch_{\mathcal{C}S}(\mathcal{E},\nabla) = ch_{\mathbb{Z}}(\mathcal{E})$ and $\epsilon_F ch_{\mathcal{C}S}(\mathcal{E},\nabla)$ sends $\varphi \in H^{2i}(BG.,\mathbb{Z}(i))$ to the value of the corresponding polynomial $\varphi_{\mathbb{C}} \in H^{2i}(BG.,\mathbb{C}) = S^i(\mathfrak{g}^*)^G$ on curvature form of ∇. For example, if ∇ is flat, the Cheeger-Simons classes live in $H^{2i-1}(X,\mathbb{C}^*)(i-1) \subset H^{2i}_{\mathcal{C}S}(X,\mathbb{Z}(i))$. A canonical morphism $\mathcal{C}S(i) \to \mathcal{D}(i)$, that comes from the embedding $F^{21}\Omega^{\cdot}_X \hookrightarrow F^i\Omega^{\cdot}_X$, sends $w_{(\mathcal{E},\nabla)\mathcal{C}S}$ to w_{calED} .

B6.4 Remark. The same formula as defines the product on Deligne and $\mathcal{U}$-complexes defines a product on Cheeger-Simons ones, so the canonical maps $\mathcal{U}_{\mathcal{E}}(\cdot) \to \mathcal{C}S(\cdot) \to D(\cdot)$ commute with products. In particular, $w_{(\mathcal{E},\nabla)\mathcal{C}S}$: $H^{2\cdot}(BG,\mathbb{Z}(\cdot)) \to H^{2\cdot}_{\mathcal{C}S}(X,\mathbb{Z}(\cdot))$ is morphism of rings.

B7 C^∞-Version

] The above constructions, as well as proofs, give a construction of Chern classes in C^∞-situation. In B.1 one should consider the $\mathbb{R}$-valued C^∞-forms, and take for G any Lie group. In B2.1, B2.2. one should assume that G is compact. In B4.1 one replaced $\mathbb{C}$ by $\mathbb{R}$; we will get, e.g., the long exact sequence $\cdots \longrightarrow H^{j-1}(X,\mathbb{R}/\mathbb{Z})(i) \to H^j(X,\mathcal{U}_{\mathcal{E}})(i) \to H^j(X,F^i\Omega^{\cdot}_X(i)) \to \cdots$. Same happens in B5, B6. The groups $H^{2i}_{\mathcal{C}S}(H,\mathbb{Z}(i))$, or $H^j_{\mathcal{D}}(X,\mathbb{Z})(j))$, are Cheeger-Simons groups of differential characters [CS], which explains their name.