MAT 127: Calculus C, Spring 2015
Solutions to Some HW12 Problems

Below you will find detailed solutions to two problems from HW12. Since the first was a WebAssign problem, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 8.7, Problem 52

Use power series to evaluate

$$\lim_{x\longrightarrow 0}\frac{1-\cos x}{1+x-\mathrm{e}^x}$$

Since for x near 0 (in fact, for all x)

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=1+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad 1-\cos x=-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\mathrm{e}^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\sum_{n=2}^{\infty}\frac{x^n}{n!}\,, \qquad 1+x-\mathrm{e}^x=-\sum_{n=2}^{\infty}\frac{x^n}{n!}\,,$$

we obtain

$$\frac{1-\cos x}{1+x-\mathrm{e}^x}=\frac{-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}{-\sum_{n=2}^{\infty}\frac{x^n}{n!}}=\frac{\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}{\sum_{n=2}^{\infty}\frac{x^n}{n!}}=\frac{-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\ldots}{\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots}$$

$$=\frac{-\frac{1}{2!}+\frac{1}{4!}x^2-\ldots}{\frac{1}{2!}+\frac{1}{3!}x+\ldots}\overset{x\longrightarrow 0}{\longrightarrow}-\frac{-1/2}{1/2}=\boxed{\text{-1}}$$

where $\ldots$ on the second line are terms involving positive powers of x, which approach 0 as $x\longrightarrow 0$.

Section 8.7, Problem 68

(a) *Let $p(x)$ be any polynomial in x and $n>0$ any positive integer. Show that*

$$\lim_{x\longrightarrow 0}x^{-n}p(x)\mathrm{e}^{-1/x^2}=0.$$

First, check this for $p(x)=1$:

$$\lim_{x\longrightarrow 0}x^{-n}\mathrm{e}^{-1/x^2}=\lim_{x\longrightarrow 0}\frac{(1/x)^n}{\mathrm{e}^{1/x^2}}=\lim_{x\longrightarrow\infty}\frac{x^n}{\mathrm{e}^{x^2}}=0;$$

the last equality follows from l'Hospital's rule, since $x^n,\mathrm{e}^{x^2}\longrightarrow\infty$, as do all derivatives of e^{x^2} (each of them is a polynomial multiplied by e^{x^2}). Thus,

$$\lim_{x\longrightarrow 0}x^{-n}p(x)\mathrm{e}^{-1/x^2}=\lim_{x\longrightarrow 0}p(x)\cdot\lim_{x\longrightarrow 0}x^{-n}\mathrm{e}^{-1/x^2}=p(0)\cdot 0=0.$$

(b) *Show that the function $f = f(x)$ given by*

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0; \end{cases}$$

is smooth and its k-th derivative is a function of the form

$$f^{\langle k \rangle}(x) = \begin{cases} x^{-n_k} p_k(x) \mathrm{e}^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

where n_k is some positive integer and $p_k(x)$ is some polynomial in x.

For $k=0$, $f^{\langle k \rangle}=f$ is indeed of the claimed form, with $n_k=0$ and $p_k(x)=1$. If $f^{\langle k \rangle}$ is of the claimed form for some $k \geq 0$ and $x \neq 0$

$$\begin{aligned} f^{\langle k+1 \rangle}(x) &= \left(x^{-n_k} p_k(x) \mathrm{e}^{-1/x^2}\right)' \\ &= -n_k x^{-n_k-1} p_k(x) \mathrm{e}^{-1/x^2} + x^{-n_k} p_k'(x) \mathrm{e}^{-1/x^2} + x^{-n_k} p_k(x) \mathrm{e}^{-1/x^2}(2/x^3) \\ &= x^{-(n_k+3)}\big((2-x^2)p_k(x) + x^3 p_k'(x)\big)\mathrm{e}^{-1/x^2}. \end{aligned}$$

For $x=0$, the derivative has to be computed directly from the definition:

$$f^{\langle k+1 \rangle}(0) = \lim_{h \longrightarrow 0} \frac{f^{\langle k \rangle}(h) - f^{\langle k \rangle}(0)}{h} = \lim_{h \longrightarrow 0} \frac{h^{-n_k} p_k(h) \mathrm{e}^{-1/h^2}}{h} = \lim_{h \longrightarrow 0} h^{-(n_k+1)} p_k(h) \mathrm{e}^{-1/h^2} = 0;$$

the last equality holds by part (a). Thus, if $f^{\langle k \rangle}$ is of the claimed form for some $k \geq 0$, then $f^{\langle k+1 \rangle}$ is of the claimed form with

$$n_{k+1} = n_k + 3, \qquad p_{k+1}(x) = (2-x^2)p_k(x) + x^3 p_k'(x).$$

This shows that $f^{\langle k \rangle}$ is of the claimed form for all k. So $f=f(x)$ is a smooth function and $f^{\langle k \rangle}(0)=0$ for all k.

(c) *Conclude that the smooth function $f(x)$ does not admit a Taylor series expansion on any neighborhood of 0 (the Taylor series of f at $x=0$ does not converge to $f(x)$ for any $x \neq 0$).*

By part (b), the Taylor expansion of $f=f(x)$ at $x=0$ would have to be

$$\sum_{n=0}^{\infty} \frac{f^{\langle n \rangle}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

Since $f(x)>0$ if $x \neq 0$, the Taylor series of f at 0 does not converge to f for any $x \neq 0$.