MAT 127: Calculus C, Spring 2015
Solutions to Some HW1 Problems

Below you will find detailed solutions to three problems from HW1. Since they were part of the WebAssign portion of this homework, your versions of these problems likely had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 7.1, Problem 3 (webassign)

(a) *For what values of r does the function $f(x)=\mathrm{e}^{rx}$ satisfy the differential equation*

$$2y''+y'-y=0\,?$$

Differentiate f and plug in into the equation:

$$f(x)=\mathrm{e}^{rx}, \qquad f'(x)=\mathrm{e}^{rx}\cdot r=r\cdot\mathrm{e}^{rx}, \qquad f''(x)=r\cdot\mathrm{e}^{rx}\cdot r=r^2\cdot\mathrm{e}^{rx}\,;$$
$$2f''+f'-f=2\cdot r^2\cdot\mathrm{e}^{rx}+r\cdot\mathrm{e}^{rx}-\mathrm{e}^{rx}=\big(2r^2+r-1\big)\mathrm{e}^{rx}=0.$$

Thus, f satisfies the differential equation if and only if

$$0=2r^2+r-1=(2r-1)(r+1);$$

so $r=1/2,-1$.

(b) *If r_1 and r_2 are the values of r you found in part (a), show that every member of the family of functions*

$$y(x)=a\,\mathrm{e}^{r_1x}+b\,\mathrm{e}^{r_2x}$$

is also a solution.

We need to show that the function

$$f(x)=a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x},$$

where a,b are any two fixed constants, solves the differential equation in part (a). So differentiate f and plug into the differential equation to compare the two sides:

$$f(x)=a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x}\,, \quad f'(x)=\frac{1}{2}a\,\mathrm{e}^{x/2}-b\,\mathrm{e}^{-x}\,, \quad f''(x)=\frac{1}{4}a\,\mathrm{e}^{x/2}+b\,\mathrm{e}^{-x}\,;$$
$$\begin{aligned}2f''+f'-f&=2\Big(\frac{1}{4}a\mathrm{e}^{x/2}+b\mathrm{e}^{-x}\Big)+\Big(\frac{1}{2}a\mathrm{e}^{x/2}-b\mathrm{e}^{-x}\Big)-\big(a\mathrm{e}^{x/2}+b\mathrm{e}^{-x}\big)\\&=\Big(\frac{1}{2}+\frac{1}{2}-1\Big)a\mathrm{e}^{x/2}+(2-1-1)b\mathrm{e}^{-x}\\&=0;\end{aligned}$$

so f indeed satisfies the differential equation.

Section 7.1, Problem 10 (webassign)

A function $y(t)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 \,.$$

(a) *What are the constant solutions of the equation?*

If the constant function $f(t) = C$ solves the equation, then

$$\begin{aligned} 0 = f'(t) &= f(t)^4 - 6f(t)^3 + 5f(t)^2 = C^4 - 6C^3 + 5C^2 \\ &= C^2\big(C^2 - 6C + 5\big) = C^2(C-1)(C-5); \end{aligned}$$

so $C = 0, 1, 5$.

(b) *For what values of y is y increasing?*

A solution y of the above differential equation is increasing if

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 > 0.$$

This is the case if

$$y^2(y^2 - 6y + 5) = y^2(y-1)(y-5) > 0,$$

i.e. if $y < 0$, $0 < y < 1$, or $y > 5$.

(c) *For what values of y is y decreasing?*

A solution y of the above differential equation is decreasing if

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^4 - 6y^3 + 5y^2 < 0.$$

This is the case if

$$y^2(y^2 - 6y + 5) = y^2(y-1)(y-5) < 0,$$

i.e. if $1 < y < 5$.

The conclusions in (a)-(c) can be restated using pictures as in Figure 1. The first diagram shows the y-axis and indicates the constant solutions by large dots. The arrows between the dots indicate whether the solutions in each interval go up and down. The second diagram shows the graphs of the constant solutions in thick lines. The arrows in each segment created by the constant solutions indicate whether the graphs of the other solutions in each segment rise or fall; these graphs must approach the two nearby constant solutions as $t \longrightarrow \pm\infty$. The first diagram makes sense only for *autonomous* first-order differential equations, i.e. equations of the form $y' = f(y)$, where y is a function of some variable t (note that there is no explicit dependence on t in the equation).

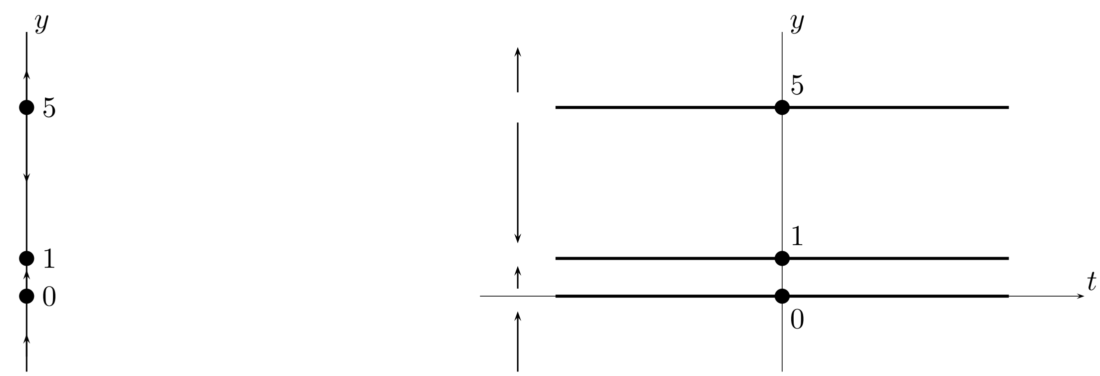


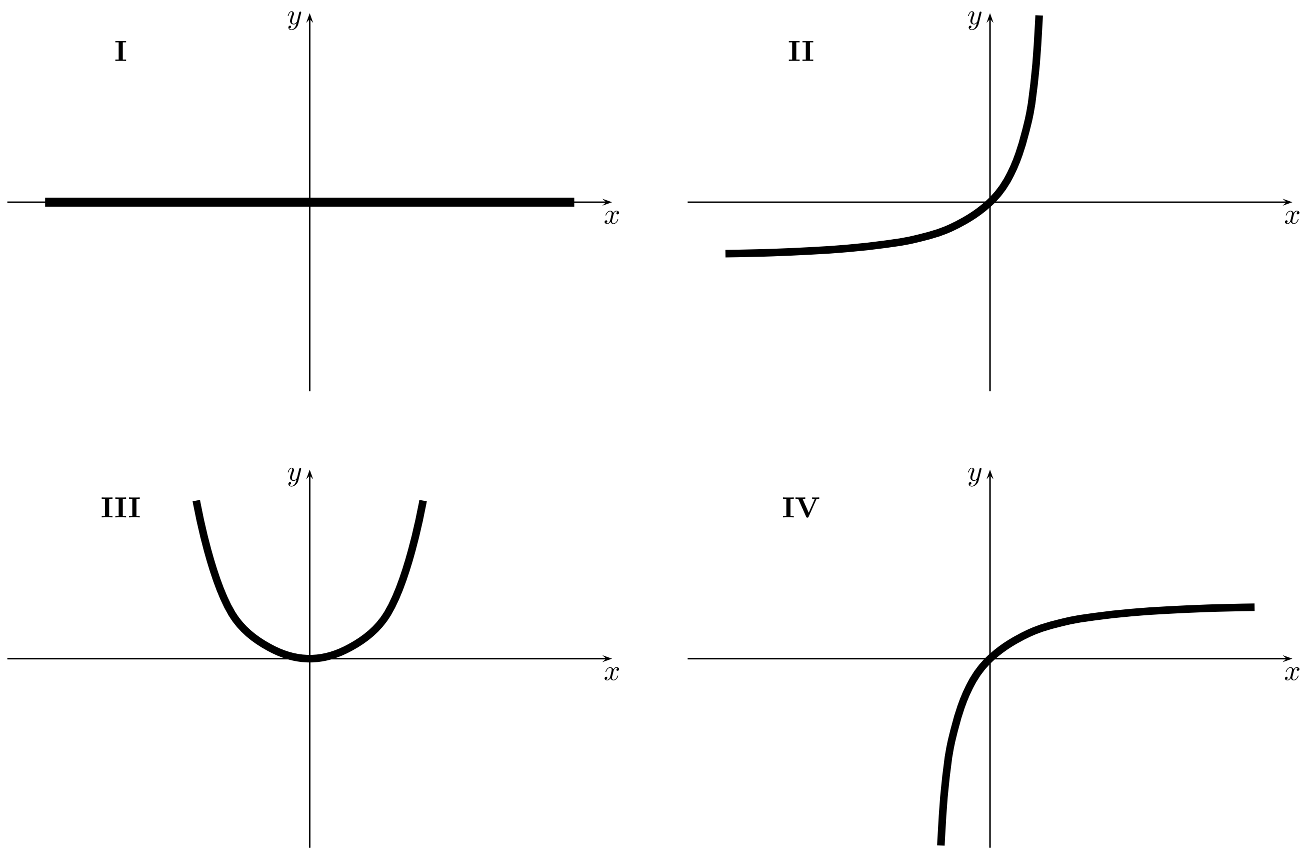
Figure 1: Diagram for Problem 10

Section 7.1, Problem 13 (webassign)

Consider the four differential equations for $y = y(x)$:

(a) $y' = x(1+y^2)$ (b) $y' = y(1+x^2)$ (c) $y' = \mathrm{e}^{x+y}$ (d) $y' = \mathrm{e}^{-x-y}$.

Each of the four diagrams below shows a solution curve for one of these equations:



Match each of the diagrams to the corresponding differential equation (the match is one-to-one).

Solution is on the next page.

diagram	I	II	III	IV
equation	b	c	a	d

Of the four diagrams, the most distinctive is **I**; it shows the graph of the constant function $y(x)=0$. Plugging in this function into the four equations, we get

$$\text{(a)}\ \ 0 \stackrel{?}{=} x(1+0^2) \qquad \text{(b)}\ \ 0 \stackrel{?}{=} 0(1+x^2) \qquad \text{(c)}\ \ 0 \stackrel{?}{=} \mathrm{e}^{x+0} \qquad \text{(d)}\ \ 0 \stackrel{?}{=} \mathrm{e}^{-x-0}\,.$$

Of these four potential equalities, only (b) is satisfied for all x; so diagram **I** must correspond to (b).

Of the three remaining graphs, the most distinctive is **III**; the slope of the graph at $(0,0)$ there is 0. This is the case for the slope of equation (a) at (0,0) (because $0(1+0^2)=0$), but the slopes for the other two remaining equations are always positive (because e^x is always positive). So diagram **III** must correspond to (a).

The two remaining graphs look rather similar. One distinguishing feature is that the graph in **II** becomes steeper as x, y increase, while the graph in **IV** less steep as x, y increase. Considering the two remaining equations, e^{x+y} becomes larger as x, y increase (thus making the slope of the graph steeper), while e^{-x-y} becomes smaller as x, y increase (thus making the slope of the graph less steep). So, diagram **II** must correspond to (c), while diagram **IV** must correspond to (d).

Alternatively, diagram **II** depicts the graph of a function with $y''>0$, while diagram **IV** depicts the graph of a function with $y''<0$. If $y=y(x)$ is any function satisfying equation (c), then by the *Chain Rule*

$$y'' = (y')' = \left(\mathrm{e}^{x+y}\right)' = \mathrm{e}^{x+y} \cdot (x+y)' = \mathrm{e}^{x+y} \cdot (1+y') = \mathrm{e}^{x+y} \cdot (1+\mathrm{e}^{x+y}) > 0,$$

because $\mathrm{e}^{x+y}>0$. On the other hand, if $y=y(x)$ is any function satisfying equation (d), then by the *Chain Rule*

$$y'' = (y')' = \left(\mathrm{e}^{-x-y}\right)' = \mathrm{e}^{-x-y} \cdot (-x-y)' = -\mathrm{e}^{-x-y} \cdot (1+y') = -\mathrm{e}^{-x-y} \cdot (1+\mathrm{e}^{-x-y}) < 0,$$

because $\mathrm{e}^{-x-y}>0$. So, **II** must correspond to (c), while **IV** must correspond to (d).