

**MAT 127: Calculus C, Fall 2009**  
**Solutions to Final Exam**

**Problem 1 (10pts)**

Determine whether each of the following sequences or series converges or not. In each case, clearly circle either **YES** or **NO**, but not both. Each correct answer is worth 2 points.

(a) the sequence  $a_n = 1 - (-1)^n$  **YES**      **NO**

The sequence keeps on jumping between 0 and 2.

(b) the sequence  $a_n = 1 + \frac{\cos(n)}{n^2}$  **YES**      **NO**

Since  $\cos(n)/n^2 \rightarrow 0$  (because  $|\cos(n)| \leq 1$ ),  $a_n \rightarrow 1$

(c) the series  $\sum_{n=1}^{\infty} \cos(1/n)$  **YES**      **NO**

Since the sequence  $\cos(1/n) \rightarrow \cos(0) = 1$ , not 0, the series diverges.

(d) the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$  **YES**      **NO**

This series is alternating,  $1/\sqrt{n} \rightarrow 0$ , and  $1/\sqrt{n} > 1/\sqrt{n+1}$ ; so the Alternating Series Test applies.

(e) the series  $\sum_{n=1}^{\infty} \frac{n + \sin n}{n^2}$  **YES**      **NO**

$(n + \sin(n))/n^2$  looks like  $n/n^2 = 1/n$ :  $\frac{(n + \sin(n))/n^2}{1/n} = \frac{n^2 + n \sin(n)}{n^2} = 1 + \frac{\sin(n)}{n} \rightarrow 1$ .

Since  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges by the  $p$ -series test ( $p=1 \leq 1$ ), so does  $\sum_{n=1}^{\infty} \frac{n + \sin n}{n^2}$  by the Limit Comparison Test. The Comparison Test can also be used:

$$|\sin(n)| \leq 1 \quad \implies \quad \frac{n + \sin(n)}{n^2} \geq \frac{n-1}{n^2} \geq \frac{n/2}{n^2} = \frac{1}{2} \cdot \frac{1}{n} \quad \text{if } n \geq 2.$$

Since  $\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$  diverges by the  $p$ -series test, so does  $\sum_{n=1}^{\infty} \frac{n + \sin n}{n^2}$  by the Comparison Test. Alternatively,

$$\sum_{n=1}^{\infty} \frac{n + \sin n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

The first series on the right-hand side diverges by the  $p$ -series test, while the last series on the right-hand side converges by the  $p$ -series test, by the Comparison Test, and the Absolute Convergence Test:

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

converges because  $0 \leq |\sin(n)|/n^2 \leq 1/n^2$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges. Since the sum of a divergent series and a *convergent* series is divergent, our series diverges.

## Problem 2 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts)  $f(x) = x^3$  around  $x = 2$

In this case, all derivatives can be computed:

$$\begin{aligned} f^{(0)}(x) = x^3 &\implies f^{(0)}(2) = 8, & f^{(1)}(x) = 3x^2 &\implies f^{(1)}(2) = 12, \\ f^{(2)}(x) = 6x &\implies f^{(2)}(2) = 12, & f^{(3)}(x) = 6 &\implies f^{(3)}(2) = 6, \end{aligned}$$

and  $f^{(n)}(x) = 0$  if  $n \geq 4$ . So by the Main Taylor Formula:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \frac{8}{0!} x^0 + \frac{12}{1!} (x-2)^1 + \frac{12}{2!} (x-2)^2 + \frac{6}{3!} (x-2)^3 \\ &= \boxed{8 + 12(x-2) + 6(x-2)^2 + (x-2)^3} \end{aligned}$$

Since this series is a sum of finitely many (four) terms, it converges for all  $x$ . So the interval of convergence is  $\boxed{(-\infty, \infty)}$ , while the radius is  $\boxed{\infty}$

*Remark:* you can check the Taylor series expansion by expanding the expression in the long box above and getting  $x^3$ .

**Grading:** statement of general Taylor formula 1pt, with  $a = 2$  3pts (*not* in addition to 1pt); vanishing of higher derivatives 1pt; computation of the remaining derivatives 2pts; final answer 1pt; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct); at least 2pts off if final answer is not a polynomial in  $(x-2)$ .

(b; 10pts)  $f(x) = \frac{x}{1-4x^2}$  around  $x = 0$

Since  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  and this power series converges if  $|x| < 1$ ,

$$\frac{x}{1-4x^2} = x \sum_{n=0}^{\infty} (4x^2)^n = x \sum_{n=0}^{\infty} 4^n (x^2)^n = x \sum_{n=0}^{\infty} 4^n x^{2n} = \boxed{\sum_{n=0}^{\infty} 4^n x^{2n+1}}$$

and this series converges whenever

$$|4x^2| < 1 \iff x^2 < 1/4 \iff -1/2 < x < 1/2;$$

so the interval of convergence is  $\boxed{(-1/2, 1/2)}$  and the radius is  $\boxed{1/2}$

**Grading:** use of correct standard power series 2pts; substitution and multiplication statement 1pt each; 3pts for simplifying to a power series in  $x$ ; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except end-points error 1pt off).

(c; bonus 10pts)  $f(x) = \frac{1}{6 - 5x + x^2}$  around  $x = 0$

First, factor out the denominator:

$$\frac{1}{x^2 - 5x + 6} = \frac{1}{(x-2)(x-3)} = \frac{1}{(x-2)} \cdot \frac{1}{(x-3)} = \frac{1}{(2-x)} \cdot \frac{1}{(3-x)}$$

Since  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$  if  $|x| < 1$ ,

$$\frac{1}{2-x} = \frac{1/2}{2/2 - x/2} = \frac{1/2}{1 - x/2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n \quad \text{if } |x/2| < 1$$

$$\frac{1}{3-x} = \frac{1/3}{3/3 - x/3} = \frac{1/3}{1 - x/3} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n \quad \text{if } |x/3| < 1.$$

Using partial fractions thus gives

$$\begin{aligned} \frac{1}{x^2 - 5x + 6} &= \frac{1}{(x-2)(x-3)} = \frac{1}{-3 - (-2)} \left( \frac{1}{x-2} - \frac{1}{x-3} \right) = -\frac{1}{x-2} + \frac{1}{x-3} \\ &= \frac{1}{2-x} - \frac{1}{3-x} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n - \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} x^n = \boxed{\sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) x^n} \end{aligned}$$

Since the  $1/(2-x)$  series converges whenever  $|x| < 2$ , while the  $1/(3-x)$  series converges whenever  $|x| < 3$ , the difference converges whenever  $|x| < 2$ . So the interval of convergence is  $\boxed{(-2, 2)}$  and the radius is  $\boxed{2}$

Alternatively, multiplication of power series can be used:

$$\begin{aligned} \frac{1}{x^2 - 5x + 6} &= \frac{1}{(2-x)} \cdot \frac{1}{(3-x)} = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{2^n} \right) \cdot \frac{1}{3} \left( \sum_{n=0}^{\infty} \frac{x^n}{3^n} \right) \\ &= \frac{1}{6} \left( 1 + \frac{x}{2} + \frac{x^2}{2^2} + \dots \right) \left( 1 + \frac{x}{3} + \frac{x^2}{3^2} + \dots \right) \\ &= \frac{1}{6} \sum_{n=0}^{\infty} \left( 1 \cdot \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right) \cdot \left(\frac{1}{3}\right)^{n-1} + \dots + \left(\frac{1}{2}\right)^{n-1} \cdot \left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)^n \cdot 1 \right) x^n \\ &= \frac{1}{6} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{2}\right)^{n+1}}{\frac{1}{3} - \frac{1}{2}} x^n = \boxed{\sum_{n=0}^{\infty} \left( \frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) x^n} \end{aligned}$$

Since the last power series is the difference of a power series convergent for  $|x| < 2$  and a power series convergent for  $|x| < 3$ , it converges for  $|x| < 2$ .

**Grading:** product decomposition of the fraction 1pt; power series for  $1/(2-x)$  or  $1/(3-x)$  2pts, with 3pts for both; 3pts for either the partial fraction or multiplication of power series computation to the answer; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except end-points error 1pt off).

**Problem 3 (20pts)**

(a; 8pts) Find the radius and interval of convergence of the power series

$$f(x) = \sum_{n=1}^{\infty} \sqrt{n}x^n.$$

To find the radius of convergence, use the Ratio Test with  $a_n = \sqrt{n}x^n$ :

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n+1}|x|^{n+1}}{\sqrt{n}|x|^n} = \sqrt{\frac{n+1}{n}} |x| = \sqrt{1 + \frac{1}{n}} |x| \longrightarrow \sqrt{1+0} |x| = |x|.$$

So the series converges if  $|x| < 1$  and diverges if  $|x| > 1$ . Thus, the radius of convergence is  $\boxed{1}$  and it remains to check convergence for  $x = \pm 1$ , i.e. whether each of the series

$$f(1) = \sum_{n=1}^{\infty} \sqrt{n}(1)^n = \sum_{n=1}^{\infty} \sqrt{n} \quad \text{and} \quad f(-1) = \sum_{n=1}^{\infty} \sqrt{n}(-1)^n = \sum_{n=1}^{\infty} (-1)^n \sqrt{n}$$

converges. Neither does because the sequences  $\sqrt{n}$  and  $(-1)^n \sqrt{n}$  do not converge to 0. So neither of the end-points is in the interval of convergence, and thus the interval of convergence is  $\boxed{(-1, 1)}$

**Grading:** radius of convergence 1pt, justification 3pts; convergence/divergence at each of the end-points and justification 1pt each; interval of convergence 1pt; at least 3pts off if the radius is found to be  $\infty$

(b; 4pts) Find  $\lim_{x \rightarrow 0} \frac{f(x) - x}{x^2}$

$$\frac{f(x) - x}{x^2} = \frac{(x + \sqrt{2}x^2 + \sqrt{3}x^3 + \dots) - x}{x^2} = \frac{\sqrt{2}x^2 + \sqrt{3}x^3 + \dots}{x^2} = \sqrt{2} + \sqrt{3}x + \dots \xrightarrow{x \rightarrow 0} \sqrt{2} + 0 = \boxed{\sqrt{2}}$$

**Grading:** expanding the series 2pts; indication of computation to the answer 2pts; use of l'Hospital rule (twice) is fine, but the 0/0 assumption must be checked each time, with 1pt off for each of the times the check is missing; if the answer is correct, at least 1pt for this part.

(c; 8pts) Find the Taylor series expansion for the function  $g = g(x)$  given by

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} du$$

around  $x=0$ . What are the radius and interval of convergence of this power series?

$$\text{Since } \frac{f(u) - u}{u^2} = u^{-2} \sum_{n=2}^{\infty} \sqrt{n} u^n = \sum_{n=2}^{\infty} \sqrt{n} u^{n-2},$$

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} du = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} u^{n-1} \Big|_{u=0}^{u=x} = \boxed{\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} x^{n-1}}$$

Since integration does not change the radius of convergence of a power series, the radius of convergence of this power series is still  $\boxed{1}$ . Since integration can add end-points to the interval of convergence, we need to check convergence of the series

$$g(1) = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} 1^{n-1} = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} \quad \text{and} \quad g(-1) = \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1} (-1)^{n-1} = \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{n-1}/\sqrt{n}}$$

Since  $\sqrt{n}/(n-1)$  looks like  $\sqrt{n}/n = 1/n^{1/2}$ ,

$$\frac{\sqrt{n}/(n-1)}{1/n^{1/2}} = \frac{n}{n-1} = \frac{1}{1-1/n} \rightarrow 1,$$

and  $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$  diverges by the  $p$ -Series test ( $p = 1/2 \leq 1$ ), the  $g(1)$  series diverges and so the end-point  $x=1$  is not in the interval of convergence. On the other hand, the  $g(-1)$  series is alternating (odd terms are positive, even terms are negative),  $1/(\sqrt{n-1}/\sqrt{n}) \rightarrow 0$ , and

$$1/(\sqrt{n-1}/\sqrt{n}) > 1/(\sqrt{n+1-1}/\sqrt{n+1})$$

(because  $\sqrt{n-1}/\sqrt{n} < \sqrt{n+1-1}/\sqrt{n+1}$ ), the  $g(-1)$  series converges by the Alternating Series Test. So the end-point  $x=-1$  is in the interval of convergence, and thus the interval of convergence is  $\boxed{[-1, 1)}$

**Grading:** integrand as a power series 2pts; power series for  $g$  1pt; radius of convergence and explanation 1pt each (use of ratio test is ok); convergence/divergence at each of the end-points and justification 1pt each (either *by Alternating Series Test* or statement of the 3 assumptions suffices; limit computation not required); no penalty for carry-over errors from (a) or inconsistencies in answers between (a) and (c)

#### Problem 4 (20pts)

Show that the following series are convergent and find their sums.

(a; 10pts)  $\sum_{n=0}^{\infty} \frac{2^n (\ln 3)^n}{n!}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=0}^{\infty} \frac{2^n (\ln 3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(2 \ln 3)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Big|_{x=2 \ln 3}.$$

Since the power series  $\sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges for all  $x$  and its sum equals  $e^x$ , the evaluation of this power

series at  $x=2 \ln 3$ , i.e. the infinite series  $\sum_{n=0}^{\infty} \frac{(2 \ln 3)^n}{n!}$ , also converges and equals

$$e^{2 \ln 3} = e^{\ln(3^2)} = 3^2 = \boxed{9}$$

You can also justify convergence using the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1} (\ln 3)^{n+1} / (n+1)!}{2^n (\ln 3)^n / n!} = \frac{2^{n+1}}{2^n} \cdot \frac{(\ln 3)^{n+1}}{(\ln 3)^n} \cdot \frac{n!}{(n+1)!} = 2(\ln 3) \frac{1}{(n+1)} \longrightarrow 0.$$

Since  $0 < 1$ , the series converges by the Ratio Test.

**Grading:** correct power series and evaluation point 3pts each; sum of power series,  $e^x$ , 1pt; rest of computation 2pts; justification of convergence 1pt; if *only* the convergence part is done, up to 5pts for part (a).

(b; 10pts)  $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} = \sum_{n=1}^{\infty} n \left( \frac{-1}{2} \right)^n = \sum_{n=1}^{\infty} n x^n \Big|_{x=-1/2}.$$

Now use standard power series to sum up the power series:

$$\begin{aligned} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if } |x| < 1 \quad \implies \quad \sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x \left( \sum_{n=0}^{\infty} x^n \right)' = x \left( \frac{1}{1-x} \right)' \\ = \frac{x}{(1-x)^2} \quad \text{if } |x| < 1. \end{aligned}$$

Since the power series  $\sum_{n=1}^{\infty} n x^n$  converges whenever  $|x| < 1$  and its sum (in those cases) equals  $x/(1-x)^2$ , the evaluation of this power series at  $x=-1/2$ , i.e. the infinite series  $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$ , also converges and equals

$$\frac{-1/2}{(1 - (-1/2))^2} = \frac{-1/2}{9/4} = \boxed{-\frac{2}{9}}$$

You can also justify convergence using the Ratio Test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \left( 1 + \frac{1}{n} \right) \cdot \frac{1}{2} \longrightarrow (1+0) \cdot \frac{1}{2} = \frac{1}{2}.$$

Since  $1/2 < 1$ , the series converges by the Ratio Test.

The series also converges by the alternating series test: it is alternating (odd terms are negative, even terms are positive),  $n/2^n \longrightarrow 0$  (since exponential grows faster than any polynomial), and  $n/2^n \geq (n+1)/2^{n+1}$ , since  $2n \geq (n+1)$ .

**Grading:** correct power series and evaluation point 2pts each; sum of power series,  $x/(1-x)^2$ , and justification 1pt each; rest of computation 2pts; mention of range of convergence for the power series and justification of convergence of infinite series 1pt each (if no mention of interval of convergence is made, but convergence is justified directly, via RT or AST, still 2pts); if *only* the convergence part is done, up to 5pts for part (b).

### Problem 5 (20pts)

Explain why each of the following series converges. Then estimate its sum to within  $1/18$  using the minimal possible number of terms, justifying your estimate; leave your answer as a simple fraction  $p/q$  for some integers  $p$  and  $q$  with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why.

(a; 10pts) 
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

This series converges because it is alternating (odd terms are positive, even terms are negative),  $1/n^2 \rightarrow 0$  as  $n \rightarrow \infty$ , and  $1/n^2 > 1/(n+1)^2$  for all  $n$ .

Since these 3 assumptions hold,

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} - \sum_{n=1}^{n=m} \frac{(-1)^{n-1}}{n^2} \right| = \left| \sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{n^2} \right| < |a_{m+1}| = \frac{1}{(m+1)^2}.$$

Since the left term needs to be at most  $1/18$ , by the inequality  $m = 4$  works. So the required estimate is

$$\sum_{n=1}^{n=m} \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^{n=4} \frac{(-1)^{n-1}}{n^2} = \frac{(-1)^0}{1^2} + \frac{(-1)^1}{2^2} + \frac{(-1)^2}{3^2} + \frac{(-1)^3}{4^2} = \frac{9 \cdot 16 - 9 \cdot 4 + 16 - 9}{3^2 \cdot 4^2} = \boxed{\frac{115}{144}}$$

This is an under-estimate for the infinite sum, because the last term used is negative.

*Remark 1:* You can justify convergence using the *Absolute Convergence Test*, since the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the *p-Series Test*.

*Remark 2:* According to the book's recipe, you need to take  $m = 4$  as done above because this is the smallest value of  $m$  for which the book's upper-bound on the remainder of the infinite series is no greater than the required precision (with  $m = 3$ , the upper-bound is  $1/(3+1)^2 = 1/16 > 1/18$ ). However, the remainder is smaller than the upper bound, so that a smaller  $m$  could still work. A sharper upper-bound for the remainder of an alternating series is given by the sum of the first *three* missing terms:

$$\left| \sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{n^2} \right| < |a_{m+1} + a_{m+2} + a_{m+3}| = \frac{1}{(m+1)^2} - \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2}.$$

For  $m = 3$ , this gives

$$\frac{1}{16} - \frac{1}{25} + \frac{1}{36} = \frac{9 \cdot 25 - 144 + 4 \cdot 25}{144 \cdot 25} = \frac{325 - 144}{18 \cdot 8 \cdot 25} = \frac{1}{18} \cdot \frac{181}{200} < \frac{1}{18}.$$

So  $m = 3$  actually works, resulting in the estimate of  $31/36$ , which is an over-estimate because the last term used is positive. On the other hand,  $m = 2$  would not work because a lower bound for the remainder of an alternating series is given by the sum of the first *four* dropped terms: in this case, this bound is

$$\frac{1}{(m+1)^2} - \frac{1}{(m+2)^2} + \frac{1}{(m+3)^2} - \frac{1}{(m+4)^2} = |a_{m+1} + a_{m+2} + a_{m+3} + a_{m+4}| < \left| \sum_{n=m+1}^{\infty} \frac{(-1)^{n-1}}{n^2} \right|.$$

If  $m = 2$ , this lower bound is  $1/9 - 1/18 \cdot 181/200 = 1/18 \cdot 219/200 > 1/18$ .

**Grading:** full justification of convergence by any argument 3pts (assumption checks are required, names of tests are not); *Alternating Series Estimation Theorem* statement for the given case 2pts; conclusion that  $m=4$  2pts; computation of finite sum 2pts (but no credit if the choice of  $m$  is not justified); under/over-estimate (depending on  $m$ ) with justification 1pt; 2pts off if the Absolute Convergence Test is used to justify convergence and the 3 assumptions for the *Alternating Series Test* (and thus Estimation Theorem) are not stated; only 1pt off (instead of 3pts) if these 3 assumptions are stated correctly, but the convergence conclusion is not made.

**Bonus:** justification that  $m=3$  works and  $m=2$  does not work 5pts each; the relevant inequality in each case must be stated to receive any bonus points; use of  $m=3$  without justification constitutes an error to be penalized out of the non-bonus points with no bonus points awarded (except possibly for the  $m=2$  part)

(b; 10pts) 
$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

This series converges by the *p-Series Test* ( $p=3 > 1$ ).

Since  $f(x) = 1/x^3 > 0$  is continuous and decreasing on  $[1, \infty)$ , by the *Remainder Estimate for the Integral Test* Theorem

$$\int_{m+1}^{\infty} \frac{1}{x^3} dx < \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{n=m} \frac{1}{n^3} = \sum_{n=m+1}^{\infty} \frac{1}{n^3} < \int_m^{\infty} \frac{1}{x^3} dx$$

Since

$$\int_m^{\infty} \frac{1}{x^3} dx = \int_m^{\infty} x^{-3} dx = \left. \frac{1}{-2} x^{-2} \right|_m^{\infty} = \frac{1}{2} m^{-2} = \frac{1}{2m^2},$$

we find that

$$\frac{1}{2(m+1)^2} < \sum_{n=m+1}^{\infty} \frac{1}{n^3} < \frac{1}{2m^2}.$$

Since we need the middle term to be at most  $1/18 = 1/(2 \cdot 3^2)$ , by the second inequality  $m=3$  works; by the first inequality  $m=2$  would not work. So the required estimate is

$$\sum_{n=1}^{n=m} \frac{1}{n^3} = \sum_{n=1}^{n=3} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} = \frac{8 \cdot 27 + 27 + 8}{8 \cdot 27} = \frac{216 + 35}{216} = \boxed{\frac{251}{216}}$$

This is an under-estimate for the infinite sum, because the finite-sum estimate is obtained by dropping only positive terms from the infinite sum.

**Grading:** justification of convergence 2pts (either *p-Series Test* or  $p=2$  suffices; use of integral test is ok if the 3 assumptions are checked); remainder bound with integrals 2pts, after integrating 1pt; conclusion that  $m=3$  2pts; computation of finite sum 2pts (but no credit if the choice of  $m$  is not justified); under/over-estimate with justification 1pt; 2pts off if the 3 assumptions for the remainder bounds are never stated; only 1pt off (instead of 2pts) if these 3 assumptions and the upper bound are stated correctly, but the convergence conclusion is not made.

**Problem 6 (20pts)**

Find the general real solution to each of the following differential equations.

(a; 6pts)  $9y'' + 4y = 0$ ,  $y = y(x)$

The associated polynomial equation is

$$9r^2 + 0r + 4 = 0 \iff r^2 = -\frac{4}{9} \iff r = \pm \frac{2}{3}i = 0 \pm \frac{2}{3}i.$$

Since the two roots are complex, the general real solution is

$$y(x) = C_1 e^{0x} \cos\left(\frac{2}{3}x\right) + C_2 e^{0x} \sin\left(\frac{2}{3}x\right) = \boxed{C_1 \cos(2x/3) + C_2 \sin(2x/3)}$$

**Grading:** associated polynomial 3pts; roots 1pts; general solution 2pts.

(b; 7pts)  $9y'' - 12y' + 4y = 0$ ,  $y = y(x)$

The associated polynomial equation is

$$9r^2 - 12r + 4 = 0 \iff (3r - 2)^2 = 0 \iff r = \frac{2}{3}.$$

Since this polynomial has a double root  $r = 2/3$ , the general solution is

$$y(x) = C_1 e^{rx} + C_2 x e^{rx} = \boxed{C_1 e^{2x/3} + C_2 x e^{2x/3}}$$

**Grading:** associated polynomial 3pts; roots 2pts; general solution 2pts.

(c; 7pts)  $9y'' - 12y' = 0$ ,  $y = y(x)$

The associated polynomial equation is

$$9r^2 - 12r = 0 \iff r(3r - 4) = 0 \iff r = 0, 4/3.$$

Since this polynomial has distinct real roots  $r_1 = 0$  and  $r_2 = 4/3$ , the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \boxed{C_1 + C_2 e^{4x/3}}$$

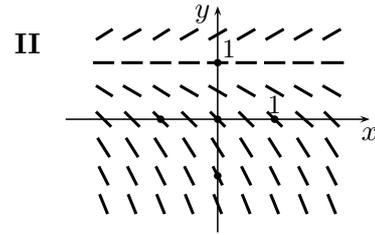
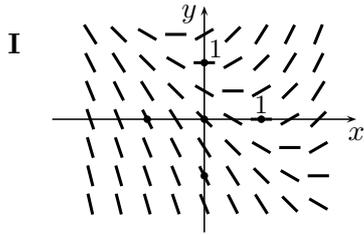
**Grading:** associated polynomial 3pts; roots 2pts; general solution (in simplified form) 2pts.

**Problem 7 (20pts)**

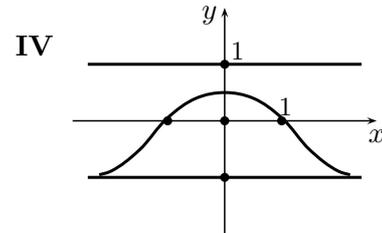
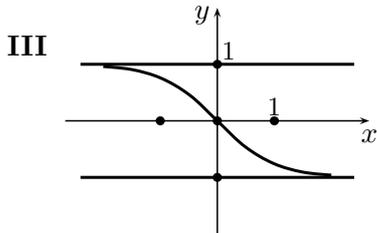
Consider the four differential equations for  $y = y(x)$ :

(a)  $y' = y - 1$ ,      (b)  $y' = y^2 - 1$ ,      (c)  $y' = x(y^2 - 1)$       (d)  $y' = x + y - 1$ .

Each of the two diagrams below shows the direction field for one of these equations:



Each of the two diagrams below shows three solution curves for one of these equations:



(so ALL three curves in diagram III are solution curves for either (a), or (b), or (c), or (d); same (?) for ALL three curves)

Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation	d	a	b	c

Explain your reasoning below.

- III, IV  $\neq$  (a),(d) because the constant function  $y = -1$  is not a solution of (a) or (d)
- III  $\neq$  (c) as a solution curve in III descends when  $y^2 < 1$  no matter whether  $x$  is positive or negative; IV  $\neq$  (b) because the slope of a solution curve in IV (increase/decrease) depends on  $x$ , not just  $y$
- II  $\neq$  (b), (c), or (d) because the slopes in II are negative for all  $y < 1$  no matter what  $x$  is
- I  $\neq$  (a) or (b) because the slopes depend on  $x$ ; I  $\neq$  (c) as the slope at  $(0,0)$  in I is  $-1$ , not  $0$

**Grading:** 1 correct answer 2pts, 2 (or 3) 5pts, 4 8pts; up to 2pts for each of 6 relevant elimination explanations (the above contains 12, half not necessary)

### Problem 8A (20pts)

A tank contains 100 liters of salt solution with 500 grams of salt dissolved in it. A salt solution containing 2g of salt per liter enters the tank at a rate of 5 liters per minute. The solution is kept thoroughly mixed and drains at a rate of 5L/min (so the volume in the tank stays constant). Let  $y(t)$  be the amount of salt in the tank, measured in grams, after  $t$  minutes.

(a; 8pts) Explain (based on the above information) why the function  $y=y(t)$  solves the initial-value problem

$$y' = 10 - \frac{y}{20}, \quad y = y(t), \quad y(0) = 500.$$

Since the initial amount of salt in the tank is 500 grams,  $y(0) = 500$ . Furthermore,  $y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t)$ , where

$$\begin{aligned} y'_{\text{in}}(t) &= (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in}} \cdot (\text{salt concentration})_{\text{in}} = 5 \cdot 2 = 10; \\ y'_{\text{out}}(t) &= (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{salt concentration})_{\text{out}}. \end{aligned}$$

Since the salt in the tank is thoroughly mixed, the outgoing salt concentration is the same as the salt concentration in the tank:

$$(\text{salt concentration})_{\text{out}} = \frac{\text{amount salt in tank}}{\text{volume in tank}} = \frac{y(t)}{100},$$

since the volume of solution in the tank is kept constant at 100 gallons. So,

$$y'_{\text{out}}(t) = 5 \cdot \frac{y(t)}{100} = \frac{y(t)}{20}.$$

It follows that  $y(t)$  is a solution to the differential equation  $y' = 10 - y/20$ .

**Grading:** mention of initial condition 1pt; computation of  $y'_{\text{in}}$  2pts and of  $y'_{\text{out}}$  3pts; remainder 2pts; mostly formulas is ok

(b; 8pts) Find the solution  $y=y(t)$  to the initial-value problem stated in (a).

First find the general solution to the differential equation. Since it is separable, writing  $y' = dy/dt$ , moving everything involving  $y$  to LHS and everything involving  $t$  to the RHS, and integrating, we obtain

$$\begin{aligned}\frac{dy}{dt} = \frac{200 - y}{20} &\iff \frac{dy}{200 - y} = \frac{dt}{20} \iff \int \frac{dy}{200 - y} = \int \frac{dt}{20} \\ &\iff -\ln|200 - y| = \frac{t}{20} + C \iff \ln|200 - y| = -\frac{t}{20} + C \\ &\iff e^{\ln|200-y|} = e^{-t/20+C} = e^C e^{-t/20} \iff |200 - y| = A e^{-t/20} \\ &\iff 200 - y = \pm A e^{-t/20} \iff y(t) = 200 + C e^{-t/20}.\end{aligned}$$

Plugging in the initial condition  $(t, y) = (0, 500)$ , we obtain

$$500 = 200 + C e^{-0/20} = 200 + C \iff C = 300.$$

So  $y(t) = 200 + 300e^{-t/20}$

**Grading:** separating variables 2pts; integration 1pt; remainder of computation to general solution 2pts; determining  $C$  2pts; final answer 1pt

(c; 4pts) How long will it take for the amount of salt in the tank to reach 300 grams?

We need to find  $t$  so that  $y(t) = 300$ :

$$\begin{aligned}y(t) = 200 + 300e^{-t/20} = 300 &\iff e^{-t/20} = \frac{1}{3} \iff -\frac{t}{20} = \ln(1/3) = -\ln 3 \\ &\iff \boxed{t = 20(\ln 3) \text{ mins}}\end{aligned}$$

**Grading:** setup 1pt; numerical solution 2pts; units 1pt

**Problem 8B (20pts)**

(a; 8pts) Show that the orthogonal trajectories to the family of curves  $xy = k$  are described by the differential equation

$$y' = \frac{x}{y}, \quad y = y(x).$$

Differentiate  $xy = k$  with respect to  $x$ , using chain rule and remembering that  $k$  is a constant:

$$x'y + xy' = 0 \quad \iff \quad xy' = -y \quad \iff \quad y' = -\frac{y}{x}$$

So our curves have slope  $y' = -y/x$  at  $(x, y)$ . The slopes of the orthogonal curves are the negative reciprocal of this; so they satisfy  $y' = -1/(-y/x) = x/y$ .

**Grading:** computation of slopes of the initial curves 4pts; the negative reciprocal statement 3pts; conclusion 1pt

(b; 6pts) Find the general solution to the differential equation stated in (a).

This equation is separable, so after writing  $y' = dy/dx$ , we can move everything involving  $y$  to LHS and everything involving  $x$  to RHS and then integrate:

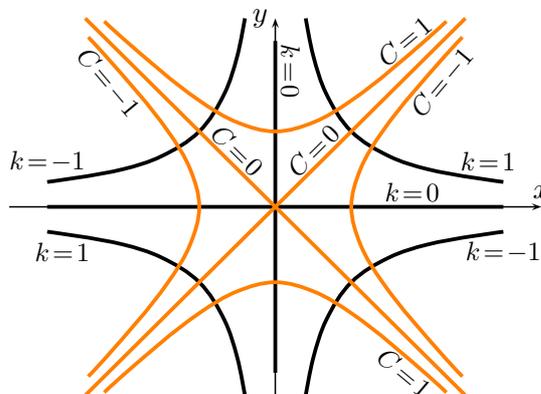
$$\frac{dy}{dx} = \frac{x}{y} \iff y \, dy = x \, dx \iff \int y \, dy = \int x \, dx \iff \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \iff \boxed{y^2 = x^2 + C}$$

**Grading:** splitting the variables 2pts; integration 2pts; simplification 2pts (answer with both square roots is ok)

(c; 6pts) Sketch at least 3 representatives of the original family of curves and at least 3 orthogonal trajectories on the same diagram; indicate clearly which is which.

Draw the above curves for different values of  $k$  and  $C$ :

- $xy=0$  consists of the two coordinate axes;  $xy=1$  is the hyperbola  $y=1/x$  in the 1st and 3rd quadrants;  $xy=-1$  is its reflection about the  $x$  or  $y$ -axis;
- $y^2=x^2+0$  consists of the lines  $y=\pm x$ ; the part of the hyperbola  $y^2=x^2+1$  above the  $x$ -axis passes through  $(0, 1)$  and rises on both sides of the  $y$ -axis, becoming asymptotic to the lines  $y=\pm x$ , while its part below the  $x$ -axis is the reflection of this about the  $x$ -axis;  $y^2=x^2-1$  is the reflection of the hyperbola  $y^2=x^2+1$  about the  $y=x$  line.



**Grading:** 1pt for each relevant curve; 2pts off if no indication is given which curves are which ( $k, C$  values not required)