MAT 127: Calculus C, Fall 2010

Solutions to Midterm II

Problem 1 (20pts)

Answer Only: *no explanation is required. Write your answer to each question in the corresponding box in the simplest possible form. No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form. In (a)-(c), assume that the limits exist.*

(a; 5pts) *Find the limit of the sequence* $a_n = \dfrac{\ln(64n^2+1) - \ln(n^2+n)}{4}$

$$\boxed{\frac{3}{2}\ln 2}$$

This is similar to 8.1 33 (on HW5). Since

$$a_n = \frac{1}{4}\ln\left(\frac{64n^2+1}{n^2+n}\right) = \frac{1}{4}\ln\left(\frac{64n^2/n^2+1/n^2}{n^2/n^2+n/n^2}\right) = \frac{1}{4}\ln\left(\frac{64+1/n^2}{1+n/n^2}\right),$$

plugging in $n=\infty$ we obtain

$$a_n \longrightarrow \frac{1}{4}\ln\left(\frac{64+1/\infty}{1+1/\infty}\right) = \frac{1}{4}\ln\left(\frac{64+0}{1+0}\right) = \frac{1}{4}\ln 2^6 = \frac{6}{4}\ln 2 = \frac{3}{2}\ln 2.$$

Grading: wrong answer 0pts; $\frac{1}{4}\ln 64$ 2pts; $\frac{1}{2}\ln 8$ 4pts; as above 5pts

(b; 5pts) *Find the limit of the sequence* $a_n = n\big(1-\mathrm{e}^{1/n}\big)$

$$\boxed{-1}$$

This is similar to 8.1 23,27 (the latter on HW5; the former was a suggested practice problem).

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty}\frac{1-\mathrm{e}^{1/n}}{1/n} = \lim_{x\longrightarrow 0}\frac{1-\mathrm{e}^x}{x} = \lim_{x\longrightarrow 0}\frac{0-\mathrm{e}^x}{1} = -\mathrm{e}^0 = -1.$$

The third equality uses l'Hospital, which is applicable here because $(1-\mathrm{e}^x), x \longrightarrow 0$ as $x \longrightarrow 0$.

Grading: wrong answer 0pts; as above 5pts

(c; 5pts) *Find the limit of the sequence*

$$\boxed{5}$$

$$\sqrt{15},\ \sqrt{15+2\sqrt{15}},\ \sqrt{15+2\sqrt{15+2\sqrt{15}}},\ \sqrt{15+2\sqrt{15+2\sqrt{15+2\sqrt{15}}}},\ \ldots$$

This is similar to 8.1 54 (on HW6). Since $a_{n+1} = \sqrt{15+2a_n}$, if this sequence converges to a, then

$$a = \sqrt{15+2a} \quad\Longrightarrow\quad a^2 = 15+2a \quad\Longrightarrow\quad a^2-2a-15=0 \quad\Longrightarrow\quad (a-5)(a+3)=0.$$

Since $a \ge 0$ by the first statement, we conclude that $a=5$.

Grading: wrong answer 0pts; as above 5pts

$$\boxed{\frac{63}{55}}$$

(d; 5pts) *Write the number* $1.1\overline{45} = 1.1454545\ldots$ *as a simple fraction*

This is similar to 8.2 36,38 (on HW6):

$$\begin{aligned}1.1\overline{45} &= 1.1 + .045 + .045\cdot\frac{1}{100} + .045\cdot\frac{1}{100^2} + \ldots\\ &= \frac{11}{10} + \frac{45/1000}{1-\frac{1}{100}} = \frac{11}{10} + \frac{45/10}{99} = \frac{11}{10} + \frac{5}{110} = \frac{121+5}{110} = \frac{63}{55}\end{aligned}$$

Grading: wrong answer 0pts; as above 5pts; $1\frac{8}{55}$ or not simplified 4pts; both issues 3pts

Problem 2 (15pts)

Determine whether each of the following sequences converges or diverges. In each case, circle your answer to the right of the question and justify it in the space provided below the question. You do not have to determine the limit if the sequence converges.

(a; 5pts) $a_n = \dfrac{(-1)^n\, n!}{n^n}$ (reminder: $n! = 1\cdot 2\cdot \ldots \cdot n$) **converge** **diverge**

Since $n!/n^n \longrightarrow 0$ (by Example 9 on p558), $a_n \longrightarrow 0$ (by Theorem 4 on p557). Alternatively, the *Ratio Test for Sequences* can be used:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = (n+1)\cdot\frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} = \frac{1}{\mathrm{e}}\,.$$

Since $1/\mathrm{e} < 1$, the sequence converges to 0.

Grading: correct answer 3pts; some explanation 2pts (anything in parenthesis not required)

(b; 5pts) $a_1 = 14,\;\; a_{n+1} = \sqrt{7a_n}$ **converge** **diverge**

This is similar to 8.1 48 (on HW6):

$$a_{n+1} = 7^{\frac{1}{2}}\cdot 7^{\frac{1}{4}}\cdot 7^{\frac{1}{8}}\ldots\cdot 7^{\frac{1}{2^n}}\cdot 14^{\frac{1}{2^n}} = 7^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}}\cdot 7^{\frac{1}{2^n}}\cdot 2^{\frac{1}{2^n}} = 7^1\cdot 2^{\frac{1}{2^n}} \longrightarrow 7\cdot 2^{\frac{1}{\infty}} = 7\cdot 1 = 7.$$

Alternatively, first show by induction that $a_{n+1}\le a_n$ for all n. This is the case for $n=1$, since

$$a_2 = \sqrt{7\cdot 14} = 7\sqrt{2} \le 7\cdot 2 = a_1.$$

If $a_n\le a_{n-1}$ for some $n\ge 2$, then $a_{n+1} = \sqrt{7a_n} \le \sqrt{7a_{n-1}} = a_n$. Thus, the sequence a_n is indeed decreasing. As it is also bounded below by 0 (successive numbers are obtained as squares), it converges (by the Monotonic Sequence Theorem on p561).

Grading: correct answer 2pts; justification up to 3pts (anything in parenthesis not required)

(c; 5pts) $a_1 = 1,\;\; a_{n+1} = \sqrt{7-3a_n}$ **converge** **diverge**

This is similar to MIIf06 2d. Write out the first few terms of this sequence: $1, 2, 1, 2, \ldots$. Since each term is determined by the preceding one, this sequence oscillates between 1 and 2.

Grading: correct answer 2pts; justification up to 3pts (but not if the wrong answer is circled)

Problem 3 (20pts)

(a; 10pts) *Determine whether the following series* **converges** or **diverges**

$$\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{4^n+1}}$$

Circle your answer above and justify it below.

This is similar to 8.2 9,20 (on HW6) and 8.3 15,20,26 (on HW7). Since

$$\frac{3^n}{\sqrt{4^n+1}} = \frac{3^n/2^n}{\sqrt{4^n+1}/\sqrt{(2^n)^2}} = \frac{(3/2)^n}{\sqrt{4^n/4^n+1/4^n}} = \frac{1.5^n}{\sqrt{1+1/4^n}} \longrightarrow \infty,$$

this series diverges by the test for divergence.

Alternatively, since the terms in this series look like $3^n/\sqrt{4^n} = (3/2)^n$, we can limit-compare this series to the geometric series $\sum 1.5^n$; this diverges, since $|1.5| \geq 1$. This limit-comparison can be made, since both series have positive terms and

$$\frac{3^n/\sqrt{4^n+1}}{(3/2)^n} = \frac{1}{\sqrt{4^n+1}/\sqrt{(2^n)^2}} = \frac{1}{\sqrt{4^n/4^n+1/4^n}} = \frac{1}{\sqrt{1+1/4^n}} \longrightarrow \frac{1}{\sqrt{1+0}} = 1.$$

Since the limit is nonzero, our series diverges because the other series does.

We can also compare to the divergent geometric series

$$\sum \frac{1}{2}\left(\frac{3}{2}\right)^n = \frac{1}{2}\sum \left(\frac{3}{2}\right)^n,$$

as both series have positive terms. Since $\sqrt{4^n+1} < \sqrt{4\cdot 4^n} = 2\cdot 2^n$, $\dfrac{3^n}{\sqrt{4^n+1}} > \dfrac{1}{2}\left(\dfrac{3}{2}\right)^n$. Thus, our series has larger terms than a divergent series, and so must also diverge.

Finally, we can also use the Ratio Test for Series:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{3^{n+1}/\sqrt{4^{n+1}+1}}{3^n/\sqrt{4^n+1}} = \frac{3^{n+1}}{3^n}\cdot\frac{\sqrt{4^n+1}}{\sqrt{4^{n+1}+1}} = 3\frac{\sqrt{4^n+1}/\sqrt{4^n}}{\sqrt{4^{n+1}+1}/\sqrt{4^n}}\\ &= 3\frac{\sqrt{4^n/4^n+1/4^n}}{\sqrt{4^{n+1}/4^n+1/4^n}} = 3\frac{\sqrt{1+1/4^n}}{\sqrt{4+1/4^n}} \longrightarrow 3\frac{\sqrt{1+0}}{\sqrt{4+0}} = \frac{3}{2}.\end{aligned}$$

Since $3/2 > 1$, the series diverges.

Grading: wrong answer 0pts regardless of explanation; correct answer 3pts; justification up to 7pts:

(A1) stating $a_n \longrightarrow \infty$ (or a_1 does not approach 0), but not showing this, 3pts (or 2pts);

(A2) choosing appropriate geom. series 2pts, informally motivating this choice is 2pts and formally 3pts (one or the other), stating that the geom. series diverges b/c $|r|>1$ 2pts

(A3) choosing appropriate geom. series 2pts, justifying this choice 3pts (loss of these 3pts if there is no appropriate factor like 1/2), stating that the geom. series diverges b/c $|r|>1$ 2pts

with 1-2pt reduction for minor computational error(s) and lack of positivity statement in A2,A3.

(b; 10pts) *Find all values of p for which the following series converges.*

$$\sum_{n=1}^{\infty} n^p \sin^2(1/n)$$

Write your answer in the box to the right and justify it below.

$$\boxed{p < 1}$$

This is similar to 8.3 29,31 (on HW7) and MIIf09 3c.

Since $\sin(1/n)$ looks like $1/n$ as $n \longrightarrow \infty$, limit-compare this series to

$$\sum_{n=1}^{\infty} n^p \cdot (1/n)^2 = \sum_{n=1}^{\infty} n^{p-2} .$$

This is a p-series; by the p-Series Test on p578, it converges if and only if $(p-2) < -1$. This limit-comparison can be made, since both series have positive terms and

$$\lim_{n\longrightarrow\infty} \frac{n^p \sin^2(1/n)}{n^p \cdot (1/n)^2} = \left(\lim_{n\longrightarrow\infty} \frac{\sin(1/n)}{(1/n)}\right)^2 = \left(\lim_{x\longrightarrow 0} \frac{\sin(x)}{x}\right)^2 = 1^2 = 1.$$

So our series converges whenever the above p-series does.

Grading: appropriate series for limit-comparison 3pts; justifying the choice 4pts (limit 3pts, positivity 1pt); simplifying to p-series, p-series statement, and final conclusion 1pt each; no penalty for carryover errors (e.g. the limit is computed incorrectly in a minor way and correct conclusions are drawn from the result, the loss of 2-3pts for the limit; higher penalties apply if the limit is way off).

Here is another approach. Since

$$\lim_{n\longrightarrow\infty} \frac{\sin(1/n)}{(1/n)} = \lim_{x\longrightarrow 0} \frac{\sin(x)}{x} = 1,$$

the sequence $n^p \cdot \sin^2(1/n) = n^{p-2}(\sin(1/n)/(1/n))^2$ does not approach 0 unless $p<2$; so if $p \geq 2$, the series diverges by the Test for Divergence. If $p < 2$, the Integral Test can be applied to the function $f(x)=x^p \sin^2(1/x)$, because it is continuous and positive whenever $x \geq 1$, and

$$f'(x) = px^{p-1}\sin^2(1/x)+x^p\cdot 2\sin(1/x)\cdot\cos(1/x)\cdot(-x^{-2}) = -x^{p-2}\sin(1/x)\big(2\cos(1/x)-px\sin(1/x)\big);$$

so $f'(x)<0$ for x sufficiently large, since $p<2$ and $\cos(1/x), x\sin(1/x) \longrightarrow 1$ as $x \longrightarrow \infty$. So, for $p<2$, the series converges if and only if the following indefinite integral does:

$$\int_1^{\infty} f(x)\mathrm{d}x = \int_1^{\infty} x^p \sin^2(1/x)\mathrm{d}x = \int_0^1 \theta^{-p}\frac{\sin^2(\theta)}{\theta^2}\mathrm{d}\theta .$$

If θ is close to 0, then $1/2 \leq \sin^2(\theta)/\theta^2 \leq 2$, and so the integral converges whenever the integral

$$\int_0^1 \theta^{-p}\mathrm{d}\theta = \begin{cases} \frac{1}{-p+1}\theta^{-p+1}\big|_0^1, & \text{if } \theta \neq 1; \\ \ln(\theta)\big|_0^1, & \text{if } \theta = 1; \end{cases}$$

does. This happens if and only if $-p+1>0$.

Grading: $x\sin(1/x) \longrightarrow 1$ as $x \longrightarrow \infty$ or equivalent 2pts; diverges for $p \geq 2$ because of TD 2pts; justification of use of IT for $p<2$ 3pts; sorting out when the resulting integral converges 3pts.

Problem 4 (20pts)

For each of the following series,

(1) *determine the corresponding sequence s_n of partial sums (sum of the first n terms);*

(2) *determine whether the series converges and if so, find it is sum.*

Simplify your answers as much as possible and justify them.

(a; 8pts) $\displaystyle\sum_{n=1}^{\infty}(-1)^n$

Since this series is $(-1)+1+(-1)+1+\ldots$, the sum of the first n terms is

$$\boxed{s_n = \begin{cases} -1, & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}}$$

Since the sequence $\{s_n\}$ diverges (it keeps on jumping between -1 and 0), the series also $\boxed{\text{diverges}}$

Another reason why this series diverges is that it is a geometric series with ratio -1 and $|-1| \geq 1$. A third reason is that the sequence $-1, 1, -1, 1, \ldots$ (which is being summed, not the sequence of partial sums) does not approach 0 (in fact, it diverges, since it keeps on jumping between -1 and 1) and thus the series diverges (by the Test for Divergence).

Grading: 4pts for s_n with minimal explanation (3pts if odd and even are switched or for $\{0,1\}$ instead of $\{-1,0\}$; 0pts for all other answers); 2pts for diverges; 2pts for justification (only if the answer is correct); anything in parenthesis not required

(b; 12pts) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-12n+5}$ *Hint:* partial fractions

The solution to the entire problem is on the next page. Below are alternative ways to show that the series $\boxed{\text{converges}}$ and get some partial credit. This is similar to 8.3 20,26 (on HW7).

Since the terms in the above series look like $\sum 1/n^2$, limit-compare the given series to $\sum 1/n^2$. The latter series is a p-series with $p=2>1$ and thus converges. This limit-comparison is applicable here because both series have positive terms for $n \geq 3$ and

$$\frac{1/(4n^2-12n+5)}{1/n^2} = \frac{1}{(4n^2-12n+5)/n^2} = \frac{1}{4-12/n+5/n^2} \longrightarrow \frac{1}{4-12/\infty+5/\infty} = \frac{1}{4}.$$

Thus, our series converges because the other series does.

We can also compare our series to the convergent p-series $\sum 1/n^2$. Since $4n^2-12n+5 > n^2$ if $n \geq 4$, $1/(4n^2-12n+5) < 1/n^2$ if $n \geq 4$. Since both series have positive terms for $n \geq 3$ and the "larger" (for $n \geq 4$) p-series converges, our series also converges.

The Integral Test can also be applied with $f(x)=1/(4x^2-12x+5)$, since this function is continuous, positive, and decreasing for $x \geq 3$. In order to compute the integral, use partial fractions from the next page and carefully take the limit of anti-derivative as $x \longrightarrow \infty$.

Grading: correct answer 2pts; justification up to 3pts (including positivity statements); *not* in addition to any points from the next page

This is similar to 8.2 30,34 (on HW6). First, partial fractions:

$$\frac{1}{4n^2-12n+5}=\frac{1}{(2n-5)(2n-1)}=\frac{1}{(-1)-(-5)}\left(\frac{1}{2n-5}-\frac{1}{2n-1}\right)=\frac{1}{4}\left(\frac{1}{2n-5}-\frac{1}{2n-1}\right)$$

This gives

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-12n+5}=\sum_{n=1}^{\infty}\frac{1}{4}\left(\frac{1}{2n-5}-\frac{1}{2n-1}\right)=\sum_{n=1}^{\infty}\frac{1}{4}\left(\frac{1}{2n-5}-\frac{1}{2(n+2)-5}\right).$$

Thus, the sequence of partial sums is given by

$$\begin{aligned}
s_n&=\frac{1}{4}\Bigg[\left(\frac{1}{2\cdot1-5}-\frac{1}{2(1+2)-5}\right)+\left(\frac{1}{2\cdot2-5}-\frac{1}{2(2+2)-5}\right)+\left(\frac{1}{2\cdot3-5}-\frac{1}{2(3+2)-5}\right)\\
&\qquad\qquad+\ldots+\left(\frac{1}{2n-5}-\frac{1}{2(n+2)-5}\right)\Bigg]\\
&=\frac{1}{4}\left[\frac{1}{2\cdot1-5}+\frac{1}{2\cdot2-5}-\frac{1}{2(n+1)-5}-\frac{1}{2(n+2)-5}\right]\\
&=\boxed{-\frac{1}{3}-\frac{1}{4}\left(\frac{1}{2n-3}+\frac{1}{2n-1}\right)=-\frac{1}{3}-\frac{n-1}{4n^2-8n+3}}
\end{aligned}$$

The second equality above is obtained by canceling the negative term in the k-th summand for $k=1,2,\ldots,n-2$ with the positive term two summands later. This leaves the positive terms in the first two summands and the negative terms in the last two summands (this also gives the right answer for $n=1$).

Since the sequence of partial sums,

$$s_n=-\frac{1}{3}-\frac{1}{4}\left(\frac{1}{2n-3}+\frac{1}{2n-1}\right)\longrightarrow-\frac{1}{3}-\frac{1}{4}\left(\frac{1}{\infty}+\frac{1}{\infty}\right)=-\frac{1}{3},$$

the series also $\boxed{\text{converges}}$ and

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-12n+5}=\boxed{-\frac{1}{3}}$$

Grading: *PFs* correct 4pts (-1pt each if 1/4 is wrong or the overall sign is reversed; -2pts if fractions have the same sign; -2pts for wrong denominators); clear indication or statement of two-step cancellation 3pts; simplifying to final answer for s_n 1pt (either expression in the first box is fine); limit of $\{s_n\}$ and justification 1pt each; series converges and sum 1pt each; no penalty for carryover errors if feasible (e.g. if 1/4 is incorrect; if PFs are badly messed up, resulting in no cancellations in s_n, likely loss of all subsequent points); if at any point, the infinite sum is split into two divergent sums, no more than 8pts for the question even if the rest is done right.

Problem 5 (25pts)

A two-species interaction is modeled by the following system of differential equations

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{500}x^2 - \frac{1}{40}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = y - \frac{1}{50}y^2 + \frac{1}{1000}xy \end{cases} \qquad (x,y) = (x(t), y(t)),$$

where t denotes time.

(a; 5pts) *Which of the following best describes the interaction modeled by this system?*

(i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Circle your answer above and justify it below.

Because of the coefficient of $-\frac{1}{40}$ in front of xy in the first equation, the x-species is hurt by the presence of the y-species (the growth rate of the former is reduced if the population of the latter is nonzero). Because of the coefficient of $+\frac{1}{1000}$ in front of xy in the second equation, the y-species benefits from the presence of the x-species. The x-species is thus the prey, and the y-species is the predator (given the above three choices).

Grading: wrong answer 0pts regardless of explanation; correct answer 3pts; two-part justification up to 2pts (anything in parenthesis not required)

(b; 12pts) *This system has 4 equilibrium (constant) solutions; find all of them and explain their significance relative to the interaction the system is modeling.* **Answer Only**.

$\boxed{(0,0)}$	no population of either species ever
$\boxed{(500,0)}$	x-population at its carrying capacity in the absence of y-population
$\boxed{(0,50)}$	y-population at its carrying capacity in the absence of x-population
$\boxed{\left(-\frac{1000}{13}, \frac{600}{13}\right)}$	meaningless, because the x-population is negative

The constant solutions are described by $(x'(t), y'(t)) = 0$. Using the above system this gives

$$\begin{cases} 0 = x\left(1 - \frac{1}{500}x - \frac{1}{40}y\right) \\ 0 = y\left(1 - \frac{1}{50}y + \frac{1}{1000}x\right) \end{cases} \iff \begin{cases} x = 0 \ \text{ or } \ 1 - \frac{1}{500}x - \frac{1}{40}y = 0 \\ y = 0 \ \text{ or } \ 1 - \frac{1}{50}y + \frac{1}{1000}x = 0 \end{cases}$$

Thus, the constant solutions are the solutions of the following systems:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} 1 - \frac{1}{500}x - \frac{1}{40}y = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ 1 - \frac{1}{50}y + \frac{1}{1000}x = 0 \end{cases} \qquad \begin{cases} 1 - \frac{1}{500}x - \frac{1}{40}y = 0 \\ 1 - \frac{1}{50}y + \frac{1}{1000}x = 0 \end{cases}$$

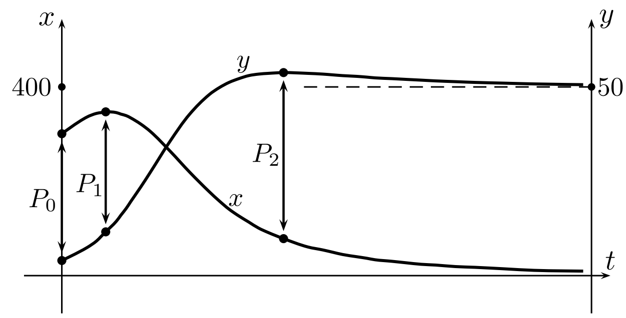
The first 3 systems quickly give the first 3 boxes above. In order to solve the last system, multiply the second equation by 2 and add the first equation to the result. This eliminates x giving us

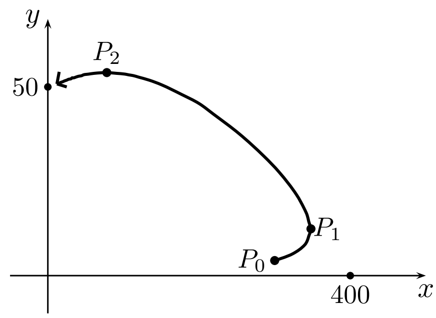
$$3 - \frac{1}{40}y - \frac{1}{25}y = 0 \quad\iff\quad 3 = \frac{5+8}{200}y \quad\iff\quad y = \frac{600}{13} \quad\Longrightarrow\quad x = -\frac{1000}{13}$$

The last statement is obtained using either of the equations in the system.

Grading: 1 correct pair 1pt, 2 3pts, 3 5pts, 4 8pts, with 1-2pts off for not simplifying fractions; significance 1pt each

(c; 8pts) *The left diagram*[1] *below shows the graphs of functions* $x = x(t)$ *and* $y = y(t)$ *so that the pair* (x, y) *solves the above system of differential equation. Sketch the corresponding (directed) phase trajectory on the right diagram below*[2]*, adding appropriate markings to the axes and indicating coordinates of whatever points possible. Explain/indicate how you make your sketch!*





Begin by copying the scale labels on the x-axis and y-axis from the left diagram to the right diagram (just 400 and 50 in this case). At time $t = 0$, the x and y-populations are about 320 and 10, respectively, giving the starting point $P_0 \approx (320, 10)$ in the phase plane. The first interesting feature in the two graphs is the peak in the x-graph (corresponding to the right-most point in the phase trajectory); at this time, the x and y-populations are about 350 and 15, respectively, giving the point $P_1 \approx (350, 15)$ in the phase plane. The second interesting feature in the two graphs is the peak in the y-graph (corresponding to the highest point in the phase trajectory); at this time, the x and y-populations are about 80 and 52, respectively, giving the point $P_2 \approx (80, 52)$ in the phase plane. After that, the x and y-populations asymptotically decline toward 0 and 50, respectively; this corresponds to the phase trajectory approaching the point $(0, 50)$ from above right in the phase plane at $t \longrightarrow \infty$, but never actually getting there.

Grading: x- and y-axes properly marked 1pt; starting point 1pt (if x- and y-axes are not properly marked, likely loss of 2pts); next feature right-most point 1pt; followed by top point 1pt; correct limiting point and direction of approach 2pts; some explanation 2pts (e.g. points marked on the left diagram and their coordinates indicated); no indication of direction 2pts off

(d; bonus 10pts, all or nothing) *Show that one of the two species eventually goes extinct, according to this model (for any initial populations).*

We will assume that $x(0), y(0) > 0$, so that both species start with positive populations (otherwise, one of them does not exist to begin with). Even in the absence of the x-species (the prey), the y-species (the predator) will approach 50 (its carrying capacity). Since the prey can only increase the population of the predator, the population of the y-species will eventually reach at least 45 (and any number below 50), whether or not there is any prey. Once the population of y is at least 45,

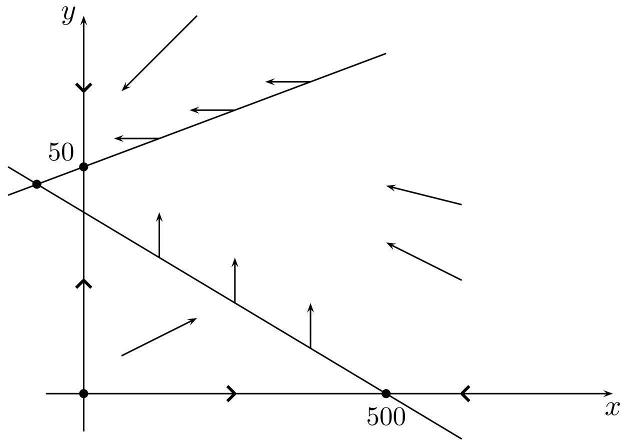
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{1}{500}x^2 - \frac{1}{40}xy \le x - \frac{45}{40}x = -\frac{1}{8}x.$$

Thus, the x-population eventually declines at least as fast as a population modeled by the exponential decay equation $P' = -\frac{1}{8}P$; the latter population satisfies $P(t) = P(0)\mathrm{e}^{-t/8}$ and declines

[1] the original left diagram did not have the line segments labeled P_0, P_1, and P_2 or the dots at their ends
[2] the original right diagram consisted of the two axes and labels x and y only

toward 0. Thus, so does the x-population (the function x never actually reaches 0, but the x-population can consist of only whole units).

Another approach is to look at the xy-plane and consider in what direction $(x(t), y(t))$ moves depending on its values. This is similar to direction field for the $\mathrm{d}y/\mathrm{d}x$-equation obtained by dividing the second equation in the system by the first (see Figures 1,2 on p542), but with arrows on the little slope segments indicating the direction of movement as t increases. In fact, we do not need the precise slopes, just the general direction (e.g. up and to the right). For example, on the x-axis (where $y=0$), $\mathrm{d}y/\mathrm{d}t=0$ according to the second equation in the system. Thus, along the x-axis, the direction of movement is horizontal; it is to the right for $x \in (0, 500)$ and to the left for $x \in (500, \infty)$, according to the first equation in the system. Similarly, on the y-axis, the direction of movement is vertical, up for $y \in (0, 50)$ and down for $y \in (50, \infty)$. According to the first equation in the system, $\mathrm{d}x/\mathrm{d}t = 0$ (the movement is vertical) at every point of the line $\frac{1}{500}x + \frac{1}{40}y = 1$ (with x- and y-intercepts of 500 and 40, respectively); according to the second equation, $\mathrm{d}y/\mathrm{d}t=0$ (the movement is horizontal) at every point of the line $-\frac{1}{1000}x + \frac{1}{50}y = 1$ (with x- and y-intercepts of -1000 and 40, respectively). The general direction of the flow does not change anywhere else. Thus, it must be up on the segment of the first line in the 1st quadrant (comparing with the direction on the y-axis) and to the left on the portion of the second line in the 1st quadrant (comparing with the direction on the x-axis); see the diagram below. There is no movement at the equilibrium points, which are indicated by dots in the sketch below. The movement in the regions formed by the above two lines and the coordinate axes is according to the nearby arrows on these lines; for example, in the upper region in the 1st quadrant the movement is down and left.



A flow beginning in the top region of the 1st quadrant can never leave it, according to the above diagram; it must thus sink into the equilibrium point $(50, 0)$. A flow beginning in the middle region of the 1st quadrant must either cross into the top region, and then sink into $(50, 0)$, or must sink directly into $(50, 0)$. A flow beginning in the bottom region of the 1st quadrant must cross into the middle region and then eventually sink into $(50, 0)$. Thus, in all cases (with y starting positive), the x-population eventually goes extinct, while the y-population approaches 50.

Grading: no partial credit; explanation must be complete on the substance in order to receive the 10 bonus points.

You can learn more about analyzing systems of differential equations in MAT 303, if you pass MAT 127 (or in MAT 308 if you get an A in MAT 127 and find it not sufficiently challenging).