RIEMANNIAN POSITIVE MASS THEOREM IN ALL DIMENSIONS IN THE PRESENCE OF LOW-CODIMENSION SINGULARITIES

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Abstract. We prove the Riemannian positive mass theorem in all dimensions for asymptotically flat L^∞-metrics with subcritical singular sets. More precisely, we consider complete asymptotically flat manifolds whose metrics are smooth away from a compact singular set of Minkowski dimension less than $n-3+\frac{2}{n}$, and whose scalar curvature is nonnegative on the regular set. We show that the ADM mass of each asymptotically flat end is nonnegative, and that the mass vanishes in some end only in the Euclidean case. For the rigidity statement, we require additionally that the Minkowski dimension of the singular set is not larger than $n-3+\frac{1}{n-1}$. This gives an asymptotically flat analogue of Schoen's codimension-three conjecture for positive scalar curvature. The proof combines a density theorem for singular asymptotically flat metrics, capacity estimates across the singular set, conformal blow-up inspired by Bi–Hao–He–Shi–Zhu [3], and a μ-bubble dimension-descent argument adapted from Brendle–Wang [6].

1. Introduction

The positive mass theorem is one of the central results in scalar curvature geometry and mathematical relativity. In its Riemannian form it asserts that a complete asymptotically flat manifold with nonnegative scalar curvature has nonnegative ADM mass, with equality only for Euclidean space. The theorem was proved by Schoen and Yau in dimensions $3 \le n \le 7$ using minimal hypersurfaces [43, 44], and by Witten in the spin case in all dimensions using the Dirac operator [53]. In the past few years there has been renewed interest in extending minimal-hypersurface methods beyond the classical regularity range. In particular, we mention the approaches of Schoen–Yau [46] and Lohkamp [36, 37] for arbitrary dimensions, as well as the work of Smale [49] and Chodosh–Mantoulidis–Schulze [13] who proved generic regularity results in dimensions 8, respectively 9 and 10, and the extension of this by Chodosh–Mantoulidis–Schulze–Wang [14] to dimension 11. Very recently, Bi–Hao–He–Shi–Zhu [3] blew up the singular set by a harmonic conformal factor to extend the Riemannian positive mass theorem up to dimension 19, and in a breakthrough Brendle–Wang [6] improved this to all dimensions by using a different conformal factor based on the distance function to the singular set and a new inductive dimensional-descent scheme. These

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latter two results make use of Lesourd–Unger–Yau's shielding principle [33] to control the additionally created complete ends. Versions of the spacetime positive mass theorem were also recently established in all dimensions in both the asymptotically flat and asymptotically hyperboloidal contexts, see [7, 25, 50].

The purpose of this paper is to prove a singular version of the Riemannian positive mass theorem in all dimensions for asymptotically flat metrics which are only L^∞ across a codimension greater than $3 - \frac{2}{n}$ singular set. Thus, an L^∞ singular set of subcritical dimension cannot be used to hide negative mass.

Theorem 1.1. *Let (M^n, g), $n \geq 3$, be a complete asymptotically flat L^∞-manifold. Suppose that the singular set $\mathcal{S}$ is compact.*

Inequality. *Assume that*

$$\dim_{\mathcal{M}} \mathcal{S} < n - 3 + \frac{2}{n}. \tag{1.1}$$

If the scalar curvature of g is nonnegative on $M^n \setminus \mathcal{S}$, then the ADM mass of each asymptotically flat end is nonnegative.

Rigidity. *Assume in addition that*

$$\dim_{\mathcal{M}} \mathcal{S} \leq n - 3 + \frac{1}{n-1}. \tag{1.2}$$

If the ADM mass of some asymptotically flat end vanishes, then M^n has only one end and there exists a homeomorphism $\Psi : M^n \to \mathbb{R}^n$ such that

$$\Psi|_{M^n \setminus \mathcal{S}} : M^n \setminus \mathcal{S} \longrightarrow \mathbb{R}^n \setminus \Psi(\mathcal{S}) \tag{1.3}$$

is a smooth Riemannian isometry. Furthermore, the metric completion of the length space $(M^n \setminus \mathcal{S}, d_g)$ is isometric, as a metric space, to Euclidean space $(\mathbb{R}^n, d_\delta)$.

Remark 1.2. *The homeomorphism Ψ constructed in the rigidity statement is actually locally bi-Lipschitz with respect to the background metric g_0. However, under the present L^∞ hypotheses, Ψ need not be C^1 across the singular set $\mathcal{S}$, as is shown by Example 5.1 below.*

The assumptions in Theorem 1.1 are essential. In particular, the L^∞-hypothesis cannot be omitted; see Definition 2.1 below for the notion of an L^∞-manifold. The negative mass Schwarzschild manifold has zero scalar curvature, a singular set of Minkowski dimension zero, and negative ADM mass, but its singularity is not L^∞: the metric is not uniformly equivalent to a smooth background metric near the singular set. Thus, the two-sided L^∞ control rules out precisely this type of degeneracy. The Minkowski dimension assumption is also sharp in spirit. Codimension-two singularities may carry conical angle defects, and such cone-type singularities can contribute a negative distributional scalar curvature concentrated along the singular set; in asymptotically flat examples this can lead to negative ADM mass despite nonnegative scalar curvature on the regular set. Therefore, one cannot expect a positive mass theorem under the sole assumption that the scalar curvature is nonnegative away from a singular set of codimension two or lower. The theorem shows that any sharp Minkowski-codimension threshold for the validity of the positive mass theorem must lie between codimensions two and $3 - \frac{2}{n}$. Determining the precise threshold remains an interesting open problem.

This result may be viewed as an asymptotically flat analogue of Schoen's codimension-three conjecture for positive scalar curvature. In the compact setting, this conjecture predicts that singular sets of codimension at least three should be invisible from the point of view of positive scalar curvature: if an L^∞-metric is smooth and has nonnegative scalar curvature away from such a set, then the singularities should not evade the usual obstructions to positive scalar curvature. A formulation of this conjecture appears, for example, in the work of Li–Mantoulidis [34, Conjecture 1.5]. Theorem 1.1 establishes a corresponding phenomenon in the asymptotically flat setting: the presence of a codimension-three L^∞-singular set does not invalidate the positive mass theorem. In other words, codimension-three L^∞-singularities cannot hide negative mass in an asymptotically flat manifold.

There is a substantial literature on positive mass theorems and scalar curvature problems for nonsmooth or singular metrics. Miao [41] proved a positive mass theorem for metrics with corners along a hypersurface under a mean-curvature jump condition, see also Shi–Tam [47]; related smoothing and Ricci-flow approaches were developed by McFeron–Szekelyhidi [40]. Lee [29] proved a positive mass theorem for Lipschitz metrics with small singular sets, while Lee–LeFloch [31] established a spin positive mass theorem for metrics with distributional scalar curvature in the class $C^0 \cap W^{1,n}_{\mathrm{loc}}$. Other singular positive mass results include work on boundary and fill-in phenomena [26, 38], incomplete manifolds [32], manifolds with structured skeleton-type singularities [34], isolated conical singularities [17], and C^0-metrics in 3-dimensions [39]. We also mention the related work of Burkhardt-Guim [8], who used Ricci flow to smooth L^∞-metrics on $\mathbb{R}^n$ with nonnegative scalar curvature away from a singular set of finite $(n-\alpha)$-dimensional Minkowski content, $\alpha > 2$, under a small L^∞-closeness assumption to the Euclidean metric.

The compact positive scalar curvature problem with L^∞-singularities has also seen important recent developments. Li–Mantoulidis [34] treated the case of dimension three, and Kazaras [28] proved desingularization results in dimension four. Dai–Sun–Wang [16] and Dai–Wang–Wang–Wei [18] established the case of isolated singularities on restricted topologies, while Wang–Xie [52] and Shi–Wang–Wu–Zhu [48] obtained further positive results for singular sets of codimension greater than or equal to $\frac{n}{2}+1$. Very recently and concurrently with the present paper, Wang–Wang–Xie [51] treated the torus case when the singular set has codimension greater than $n-3+\frac{2}{n}$, while Bi–Zhu [4] obtained an analogous result for enlargeable manifolds under a similar hypothesis involving the Assouad dimension. In contrast, Cecchini–Frenck–Zeidler [9] showed that, in high dimensions, there are counterexamples to Schoen's conjecture with point singularities. These compact results show that the codimension-three question is quite subtle.

On the other hand, as Theorem 1.1 shows, these subtleties disappear when the conjecture is considered in the noncompact context of the positive mass theorem.

The proof of Theorem 1.1 involves three primary components. The first is a density and deformation argument, that perturbs the metric to one of the same type of singularities but with strictly positive scalar curvature away from the singular set, harmonic asymptotics, and an ADM mass that is nearly unchanged. The codimension-$(3-\frac{2}{n})$ assumption enters analytically partially through capacity, namely, compact sets of Minkowski dimension less than $n-2$ have vanishing $W^{1,2}$-capacity. This permits integration by parts across $\mathcal{S}$, yields the global Sobolev inequality, and allows one to solve the elliptic equations needed for the deformation. The second component consists of establishing a generalization of the main positive mass-type theorem proven by Brendle–Wang [6, Theorem 1.5], so that it applies to what we call weak n-data sets. Here, the notion of weak n-data set primarily differs from the Brendle–Wang n-data set concept by a weakening of the weighted scalar curvature integral inequality. Lastly, the third component of the proof is inspired by the work of Bi–Hao–He–Shi–Zhu [3] and Brendle–Wang [6]. In particular, using the codimension threshold $3-\frac{2}{n}$, we conformally blow up the singular set with Green's functions, producing a complete smooth manifold $M^n \setminus \mathcal{S}$ with the chosen asymptotically flat end preserved and satisfying a pointwise weighted scalar curvature inequality. A stable μ-bubble is then constructed, which serves as a lower-dimensional asymptotically flat space carrying an integral weighted scalar curvature inequality, placing it within the context of weak $(n-1)$-data sets and their positive mass-type theorem. The rigidity statement is obtained by extending the standard splitting argument to show that the regular part is flat, while L^∞-control across the singular set together with the capacity property rule out nontrivial singular behavior. However, difficulties arise due to a lack of uniform Hessian control for harmonic functions near the singular set, and to overcome this, a stronger codimension threshold of $3-\frac{1}{n-1}$ is needed for our approach to rigidity.

The paper is organized as follows. In Section 2 we develop the elliptic theory for L^∞-metrics with codimension-three singular sets, including integration by parts across $\mathcal{S}$, a global Sobolev inequality, and the density theorem producing harmonically asymptotically flat metrics. In Section 3 we blow up the singular set and construct the μ-bubbles used in the dimension-reduction argument. Section 4 formulates and establishes the generalized positive mass-type inequality for weak n-data sets, following Brendle–Wang [7], and this is applied to obtain the inequality portion of Theorem 1.1. Finally, in Section 5 we prove rigidity and complete the proof of the singular positive mass theorem.

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2. A Density Theorem for L^∞-Metrics

In this section we develop the theory of elliptic PDE on L^∞-asymptotically flat manifolds with low-codimension singularities. This is then used to achieve perturbations to positive scalar curvature and harmonic asymptotics. We begin with several definitions.

Definition 2.1. *Let M^n be a smooth manifold equipped with a smooth Riemannian metric g_0. A measurable symmetric $(0,2)$-tensor g is called an L^∞-*metric *with respect to g_0 if every point $x \in M^n$ admits a neighborhood V_x and a constant $\Lambda_x \geq 1$ such that*

$$\Lambda_x^{-1} g_0 \leq g \leq \Lambda_x g_0 \qquad a.e. \text{ on } V_x \tag{2.1}$$

as quadratic forms. We say that (M^n, g) is a complete L^∞-Riemannian manifold *if g_0 is complete and (2.1) holds with a uniform constant $\Lambda \geq 1$. The* regular set *of an L^∞-metric g consists of those points having a neighborhood on which g is smooth (admits a smooth representative), and the* singular set *is the complement of the regular set.*

Definition 2.2. *A complete L^∞-Riemannian manifold (M^n, g), with $n \geq 3$, is called* asymptotically flat *if there exists a compact set $\mathcal{K} \subset M^n$ such that the following properties hold.*

(1) The complement $M^n \setminus \mathcal{K}$ has finitely many components $\{\mathcal{E}_i\}_{i=1}^k$, called ends, each of which is diffeomorphic to $\mathbb{R}^n \setminus \overline{B}_1$.

(2) For each i the metric $g|_{\mathcal{E}_i}$ is smooth, and after pulling back by the diffeomorphism $\mathbb{R}^n \setminus \overline{B}_1 \to \mathcal{E}_i$, it asymptotes to the Euclidean metric δ in Cartesian coordinates with decay

$$|\mathring{\nabla}^l (g|_{\mathcal{E}_i} - \delta)|_\delta = O(r^{-q-l}), \qquad l = 0, 1, 2, \tag{2.2}$$

for some $q > \frac{n-2}{2}$, where $\mathring{\nabla}$ denotes covariant differentiation with respect to δ.

(3) For each i the scalar curvature of the end is integrable, $R_g \in L^1(\mathcal{E}_i)$.

Remark 2.3. *By smoothly deforming the background metric g_0, it may be assumed that g_0 agrees with the Euclidean metric in each asymptotically flat end. Thus, we may take the background manifold (M^n, g_0) itself to be asymptotically flat in Definition 2.2.*

Remark 2.4. *By definition, the singular set of an L^∞-Riemannian manifold is closed. Furthermore, in the asymptotically flat context it is contained within a compact set $\mathcal{K}$, and hence the singular set for these type of manifolds is necessarily compact.*

For each asymptotically flat end, the *ADM mass* is defined by

$$m = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \sum_{i,j=1}^{n} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \nu^j \, dA_\delta, \tag{2.3}$$

where ν and dA_δ are respectively the Euclidean outward unit normal and hypersurface measure of the coordinate sphere S_r. Under the above decay assumptions, and with the corresponding integrability of the scalar curvature, this quantity is independent of the choice of asymptotically flat coordinates and hence is a well-defined geometric invariant of the end, as shown by Bartnik and Chruściel [1, 15].

We next recall the class of asymptotically flat metrics with harmonic asymptotics. This special form is useful because it simplifies the asymptotics, and the leading coefficient in the asymptotic expansion directly determines the ADM mass.

Definition 2.5. *An end $(\mathcal{E}, g)$ of an asymptotically flat (L^∞)-Riemannian manifold is called* harmonically asymptotically flat *if the following decay*[1] *holds in the asymptotic coordinate system*

$$g - \left(1 + \frac{2m}{(n-2)r^{n-2}}\right)\delta \in C^{2;\alpha}_{-\mathring{q}}(\mathcal{E}), \tag{2.4}$$

for some $m \in \mathbb{R}$, $\mathring{q} > n-2$, and $\alpha \in (0,1)$. In this case, m is the ADM mass of $\mathcal{E}$.

Remark 2.6. *It will be convenient in some places to extend the definition of harmonically asymptotically flat to accommodate 2-dimensional ends, by removing the mass term from (2.4). In this situation the mass is defined to be zero.*

The main result of this section is the following density theorem, which generalizes a well-known result in the setting of smooth asymptotically flat manifolds [5, Lemma 1].

Theorem 2.7. *Let (M^n, g) be a complete asymptotically flat L^∞-manifold. Assume that the singular set $\mathcal{S}$ satisfies the Minkowski dimension upper bound $\dim_{\mathcal{M}} \mathcal{S} < n-2$. Suppose that $R_g \geq 0$ on $M^n \setminus \mathcal{S}$. Then for each $\varepsilon > 0$ there exists a complete asymptotically flat L^∞-metric g' on M^n with the following properties.*

1. *It holds that $\|g' - g\|_{L^\infty(M^n, g_0)} < \varepsilon$.*
2. *g' is smooth on $M^n \setminus \mathcal{S}$ and $R_{g'} > 0$ on $M^n \setminus \mathcal{S}$.*
3. *On each asymptotically flat end, the new metric g' is harmonically asymptotically flat.*
4. *On each asymptotically flat end, if m' and m denote the respective ADM masses of g' and g, then $|m' - m| < \varepsilon$.*

2.1. **Laplace-Beltrami operator.** When the singular set $\mathcal{S}$ is of Minkowski codimension greater than two, it has vanishing $W^{1,2}$-capacity. In particular, it is invisible to $W^{1,2}$-analysis in the sense that $W^{1,2}(M^n \setminus \mathcal{S}, g)$ agrees with $W^{1,2}(M^n, g)$, and integration by parts is not disrupted.

Proposition 2.8 (Integration by parts across a codimension-three singular set)**.** *Let (M^n, g) be an L^∞-manifold. If the singular set $\mathcal{S}$ is compact and satisfies the Minkowski dimension upper bound $\dim_{\mathcal{M}} \mathcal{S} < n-2$, then the singular set has*

[1] See [30, Appendix A] for the definition of weighted Hölder spaces.

vanishing $W^{1,2}$-capacity with respect to g. In particular consider $u \in W^{1,2}(M^n, g)$, and let $v \in C^\infty(M^n \setminus \mathcal{S})$ have compact support in M^n in addition to

$$v \in W^{1,2}(M^n, g), \qquad |\nabla v|_g \in L^\infty(M^n \setminus \mathcal{S}), \qquad \Delta_g v \in L^2(M^n \setminus \mathcal{S}), \tag{2.5}$$

then

$$\int_{M^n \setminus \mathcal{S}} u\, \Delta_g v \, dV_g = -\int_{M^n \setminus \mathcal{S}} \langle \nabla u, \nabla v \rangle_g \, dV_g. \tag{2.6}$$

Proof. Let g_0 be the background metric. Since the Minkowski dimension of $\mathcal{S}$ exists and is of codimension greater than two, for every δ satisfying

$$\max\{0, \dim_{\mathcal{M}} \mathcal{S} + 3 - n\} < \delta < 1 \tag{2.7}$$

there exists a constant C_δ such that, for all sufficiently small $r > 0$,

$$\mathrm{Vol}_{g_0}\big(N_r(\mathcal{S})\big) \le C_\delta r^{3-\delta}. \tag{2.8}$$

Here $N_r(\mathcal{S})$ denotes the r-neighborhood of $\mathcal{S}$ with respect to g_0.

Choose smooth cutoff functions $\chi_r \in C_c^\infty(M^n)$ satisfying

$$0 \le \chi_r \le 1, \qquad \chi_r = 1 \quad \text{on } N_r(\mathcal{S}), \qquad \operatorname{supp} \chi_r \subset N_{2r}(\mathcal{S}), \tag{2.9}$$

and

$$|\nabla \chi_r|_{g_0} \le \frac{C}{r}. \tag{2.10}$$

Since g and g_0 are uniformly equivalent on a fixed compact neighborhood of $\mathcal{S}$, there is a constant C, independent of r, such that

$$|\nabla \chi_r|_g^2 \, dV_g \le C |\nabla \chi_r|_{g_0}^2 \, dV_{g_0}. \tag{2.11}$$

Therefore

$$\int_{M^n} |\nabla \chi_r|_g^2 \, dV_g \le \frac{C}{r^2} \mathrm{Vol}_{g_0}\big(N_{2r}(\mathcal{S})\big) \le C_\delta r^{1-\delta}. \tag{2.12}$$

Since $\delta < 1$, it follows that

$$\|\nabla \chi_r\|_{L^2(M^n, g)} \to 0 \qquad \text{as } r \to 0. \tag{2.13}$$

Thus $\mathcal{S}$ has vanishing $W^{1,2}$-capacity.

Set $\eta_r = 1 - \chi_r$. Then $\eta_r u$ vanishes in a neighborhood of $\mathcal{S}$. Hence, by integration by parts on the smooth manifold $M^n \setminus \mathcal{S}$, and using the compact support of v, we obtain

$$\begin{aligned} \int_{M^n \setminus \mathcal{S}} \eta_r u\, \Delta_g v \, dV_g &= -\int_{M^n \setminus \mathcal{S}} \langle \nabla(\eta_r u), \nabla v \rangle_g \, dV_g \\ &= -\int_{M^n \setminus \mathcal{S}} \eta_r \langle \nabla u, \nabla v \rangle_g \, dV_g + \int_{M^n \setminus \mathcal{S}} u \langle \nabla \chi_r, \nabla v \rangle_g \, dV_g. \end{aligned} \tag{2.14}$$

We now let $r \to 0$. Since $\eta_r \to 1$ almost everywhere and $u\, \Delta_g v \in L^1(M^n \setminus \mathcal{S})$, we have

$$\int_{M^n \setminus \mathcal{S}} \eta_r u\, \Delta_g v \, dV_g \to \int_{M^n \setminus \mathcal{S}} u\, \Delta_g v \, dV_g. \tag{2.15}$$

Similarly,

$$\int_{M^n\setminus\mathcal{S}} \eta_r \langle \nabla u, \nabla v\rangle_g \, dV_g \to \int_{M^n\setminus\mathcal{S}} \langle \nabla u, \nabla v\rangle_g \, dV_g. \tag{2.16}$$

The remaining cutoff error satisfies

$$\left| \int_{M^n\setminus\mathcal{S}} u \langle \nabla \chi_r, \nabla v\rangle_g \, dV_g \right| \leq \|u\|_{L^2(M^n,g)} \|\nabla v\|_{L^\infty(M^n\setminus\mathcal{S},g)} \|\nabla \chi_r\|_{L^2(M^n,g)}. \tag{2.17}$$

Since

$$\|\nabla \chi_r\|_{L^2(M^n,g)} \to 0, \tag{2.18}$$

this error term vanishes. Therefore

$$\int_{M^n\setminus\mathcal{S}} u \, \Delta_g v \, dV_g = -\int_{M^n\setminus\mathcal{S}} \langle \nabla u, \nabla v\rangle_g \, dV_g. \tag{2.19}$$

□

2.2. **Existence of positive solutions.** Throughout the remainder of this section, it will be assumed for convenience that M^n has a single end $\mathcal{E}$. The case of additional ends may be treated with minor modifications. Consider the equation with prescribed asymptotics

$$\Delta_g u - \mathbf{c} u = 0 \text{ on } M^n, \qquad u \to 1 \text{ as } r \to \infty, \tag{2.20}$$

with $\mathbf{c} \in C^{0,\alpha}_{-n-q_0}(\mathcal{E}) \cap C^\infty(M^n \setminus \mathcal{S}) \cap L^\infty(M^n)$ for some $\alpha \in (0,1)$ and $q_0 > 0$. To study this problem, we record the following Sobolev inequality.

Lemma 2.9. *Let (M^n, g) be a complete asymptotically flat L^∞-manifold. Assume that the singular set $\mathcal{S}$ satisfies the Minkowski dimension upper bound $\dim_{\mathcal{M}} \mathcal{S} < n-2$. Then there exists a constant $C_* > 0$, depending only on the geometry of (M^n, g), such that for every compactly supported function $u \in W^{1,2}(M^n, g)$, one has*

$$\left(\int_{M^n} |u|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} \leq C_* \int_{M^n} |\nabla u|_g^2 \, dV_g. \tag{2.21}$$

Proof. Since the background metric g_0 is complete, smooth, and asymptotically flat, the usual Sobolev inequality holds on (M^n, g_0); see [43, Lemma 3.1] for $n = 3$ and note that the same proof holds in all dimensions. In particular, by the uniform equivalence between g and g_0, the inequality holds with respect to g for all $u \in C^\infty_c(M^n \setminus \mathcal{S})$. Since $\mathcal{S}$ is (Minkowski) codimension greater than two, it has vanishing $W^{1,2}$-capacity by Proposition 2.8. Thus, $C^\infty_c(M^n \setminus \mathcal{S})$ is dense inside $W^{1,2}(M^n, g)$, yielding the desired result. □

Theorem 2.10. *Let (M^n, g) be a complete asymptotically flat L^∞-manifold with asymptotic end $\mathcal{E}$. Assume that the singular set $\mathcal{S}$ satisfies the Minkowski dimension upper bound $\dim_{\mathcal{M}} \mathcal{S} < n-2$. Suppose that $\mathbf{c} \in C^{0,\alpha}_{-n-q_0}(\mathcal{E}) \cap C^\infty(M^n \setminus \mathcal{S}) \cap L^\infty(M^n)$ and satisfies*

$$\left(\int_{M^n} |\mathbf{c}_-|^{\frac{n}{2}} dV_g \right)^{\frac{2}{n}} \leq \frac{1}{2C_*}, \qquad \mathbf{c}_- := \min\{0, \mathbf{c}\}, \tag{2.22}$$

where C_* is a constant from Lemma 2.9. Then there is a positive solution $u \in C^\infty(M^n \setminus \mathcal{S}) \cap W^{1,2}(M^n, g) \cap L^\infty(M^n)$ of (2.20) such that $u - (1 + \frac{\mathcal{C}}{r^{n-2}}) \in C^{2,\alpha}_{2-n-q'}(\mathcal{E})$, where $q' = \min\{1, q_0\}$ and

$$\mathcal{C} = -\frac{1}{(n-2)\omega_{n-1}} \int_{M^n} \mathbf{c} u \, dV_g. \tag{2.23}$$

Moreover, u is bounded below by a positive constant in the essential-infimum sense, that is

$$\operatorname*{ess\,inf}_{M^n} u > 0. \tag{2.24}$$

Proof. Let $\{\Omega_i\}_{i=1}^\infty$ be an exhaustion of M^n by precompact connected open sets, each of which has a boundary $\partial\Omega_i \subset \mathcal{E}$ that is a coordinate sphere. For each i, the kernel of $\Delta_g - \mathbf{c}$ in $W^{1,2}_0(\Omega_i, g)$ is trivial. To see this, observe that if a function $w \in \ker(\Delta_g - \mathbf{c}) \cap W^{1,2}_0(\Omega_i, g)$ then utilizing Lemma 2.9, Hölder's inequality, and the notion of weak solution produces

$$\begin{aligned} \int_{\Omega_i} |\nabla w|_g^2 dV_g = -\int_{\Omega_i} \mathbf{c} w^2 dV_g &\leq \left(\int_{M^n} |\mathbf{c}_-|^{\frac{n}{2}} dV_g\right)^{\frac{2}{n}} \left(\int_{M^n} w^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} \\ &\leq \frac{1}{2C_*}\left(\int_{M^n} w^{\frac{2n}{n-2}} dV_g\right)^{\frac{n}{n-2}} \leq \frac{1}{2}\int_{M^n} |\nabla w|_g^2 dV_g = \frac{1}{2}\int_{\Omega_i} |\nabla w|_g^2 dV_g, \end{aligned} \tag{2.25}$$

and hence $w = 0$ in Ω_i. It follows by elliptic theory that there is then a unique weak solution $w_i \in C^\infty(\Omega_i \setminus \mathcal{S}) \cap W^{1,2}_0(\Omega_i, g)$ to the Dirichlet problem

$$\Delta_g w_i - \mathbf{c} w_i = \mathbf{c} \quad \text{in } \Omega_i, \qquad w_i = 0 \quad \text{on } \partial\Omega_i. \tag{2.26}$$

We will now make uniform estimates for the sequence $\{w_i\}_{i=1}^\infty$. The same manipulations as in (2.25) yield

$$\begin{aligned} \int_{M^n} |\nabla w_i|_g^2 dV_g &= -\int_{M^n} \mathbf{c} w_i^2 dV_g - \int_{M^n} \mathbf{c} w_i dV_g \\ &\leq \frac{1}{2C_*}\left(\int_{M^n} w_i^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} + \left(\int_{M^n} \mathbf{c}^{\frac{2n}{n+2}} dV_g\right)^{\frac{n+2}{2n}} \left(\int_{M^n} w_i^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{2n}}. \end{aligned} \tag{2.27}$$

This together with Lemma 2.9 implies that

$$\left(\int_{M^n} w_i^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{2n}} \leq 2C_* \left(\int_{M^n} \mathbf{c}^{\frac{2n}{n+2}} dV_g\right)^{\frac{n+2}{2n}}. \tag{2.28}$$

Note that the right-hand side is finite since $\mathbf{c} \in C^{0,\alpha}_{-n-q_0}(\mathcal{E}) \cap L^\infty(M^n)$.

Moser iteration then provides a uniform L^∞ bound on any fixed compact subset of M^n. Note that this holds even across the singular set, since the coefficients of Δ_g, when expressed in divergence form, are controlled in L^∞ near $\mathcal{S}$. We may then apply Schauder estimates to obtain uniform $C^{2,\alpha}$ bounds on compact subsets of $M^n \setminus \mathcal{S}$. Therefore, a diagonal argument may be used to extract a $C^{2,\alpha}$-convergent subsequence (with notation unchanged for convenience) $w_i \to w$. This limit solves the equation of (2.26) on M^n, and by elliptic regularity as well as the estimates above $w \in C^\infty(M^n \setminus \mathcal{S}) \cap W^{1,2}(M^n, g) \cap L^\infty(M^n)$. Furthermore, the basic asymptotic analysis of [30, Corollary A.38] shows that $w - \frac{\mathcal{C}}{r^{n-2}} \in C^{2,\alpha}_{2-n-q'}(\mathcal{E})$, for some constant $\mathcal{C}$ where $q' = \min\{1, q_0\}$. By setting $u = 1 + w$, we find that this function satisfies equation (2.20) along with all other desired properties, except perhaps positivity.

To show positivity, let $u_- = \min\{0, u\}$ and observe that $u_- \in W^{1,2}_0(\Omega_i, g)$. Using the notion of weak solution and u^- as a test function produces

$$\int_{\Omega_i} \left(\langle \nabla u, \nabla u_- \rangle_g + \mathbf{c} u u_-\right) dV_g = 0. \tag{2.29}$$

We may then replace u with u_- in this equation, and ignore the positive part of $\mathbf{c}$ to find

$$\int_{\Omega_i} |\nabla u_-|_g^2 dV_g \leq \int_{\Omega_i} |\mathbf{c}_-| u_-^2 dV_g. \tag{2.30}$$

The same strategy as in (2.25) then implies that u_- vanishes in $\Omega_i \setminus \mathcal{S}$. Since this sequence of domains is an exhaustion, we obtain $u \geq 0$ on $M^n \setminus \mathcal{S}$. The weak Harnack inequality [21, Theorem 8.18] then shows that u must be strictly positive away from the singular set. In fact, since the equation holds weakly across $\mathcal{S}$ and g is uniformly equivalent to a smooth background metric, the same weak Harnack inequality applies on balls intersecting $\mathcal{S}$ as well. Hence u has a representative which is strictly positive on M^n, and

$$\operatorname*{ess\,inf}_B u > 0 \tag{2.31}$$

for every relatively compact ball $B \subset M^n$. Together with the asymptotic condition $u \to 1$, this gives a global positive lower bound for u in the essential-infimum sense.

Lastly, to obtain the monopole expression let $\phi_r \in C^\infty_c(M^n)$ be smooth cut-off functions such that $\phi_r = 1$ on M^n_r and $\phi_r = 0$ on $M^n \setminus M^n_{2r}$ with $|\nabla \phi_r|_g \leq 2/r$, in which M^n_r denotes the bounded component of $M^n \setminus S_r$ where $S_r \subset \mathcal{E}$ is the r-coordinate sphere. Then using the definition of weak solution and integrating by parts produces

$$\begin{aligned} \int_{M^n} \mathbf{c} u \phi_r dV_g = -\int_{M^n} \langle \nabla u, \nabla \phi_r \rangle_g dV_g &= \int_{M^n_{2r} \setminus M^n_r} \phi_r \Delta_g u dV_g + \int_{S_r} \nu(u) dA_g \\ &= \int_{M^n_{2r} \setminus M^n_r} \phi_r \mathbf{c} u dV_g - (n-2)\omega_{n-1}\mathcal{C} + o(1), \end{aligned} \tag{2.32}$$

where ν is the unit normal pointing towards infinity. Since $\mathbf{c} = O(r^{-n-q_0})$, the bulk integral on the right-hand side tends to zero as $r \to \infty$, and the desired result follows. $\square$

2.3. **Proof of Theorem 2.7.**

Proposition 2.11. *Assume the setting and notation of Theorem 2.7. Then for each $\varepsilon > 0$ there exists a complete asymptotically flat L^∞-metric $\hat{g}$ on M^n with the following properties.*

(1) There exists a uniform constant $c > 0$ such that $\|\hat{g} - g\|_{L^\infty(M^n, g_0)} \leq c\varepsilon$.

(2) *$\hat{g}$ is smooth on $M^n \setminus \mathcal{S}$ and $R_{\hat{g}} > 0$ on $M^n \setminus \mathcal{S}$.*

(3) *If m and $\hat{m}$ denote the ADM masses of g and $\hat{g}$ on $\mathcal{E}$, then $\hat{m} = m + 2\varepsilon$.*

Proof. Choose a negative function $\hat{\mathbf{c}} \in C^\infty(M^n) \cap C^{0,\alpha}_{-n-q_0}(\mathcal{E})$ for some $\alpha \in (0,1)$ and $q_0 > 0$, with $\|\hat{\mathbf{c}}\|_{L^{n/2}(M^n)} \le \frac{1}{2C_*}$, and consider the following equation with prescribed asymptotics

$$\Delta_g u - \hat{\mathbf{c}} u = 0 \quad \text{on } M^n, \qquad u(x) \to 1 \quad \text{as } |x| \to \infty. \tag{2.33}$$

We now apply Theorem 2.10 to find a positive solution $u \in C^\infty(M^n \setminus \mathcal{S}) \cap W^{1,2}(M^n, g) \cap L^\infty(M^n)$ of (2.33) with $u - (1 + \frac{\hat{\mathcal{C}}}{r^{n-2}}) \in C^{2,\alpha}_{2-n-q'}(\mathcal{E})$, where

$$\hat{\mathcal{C}} = -\frac{1}{(n-2)\omega_{n-1}} \int_{M^n} \hat{\mathbf{c}} u \, dV_g > 0. \tag{2.34}$$

For each $\varepsilon \in (0, \hat{\mathcal{C}})$ set

$$f = \hat{\mathcal{C}}^{-1}\varepsilon u + (1 - \hat{\mathcal{C}}^{-1}\varepsilon), \qquad \hat{g} = f^{\frac{4}{n-2}} g. \tag{2.35}$$

This defines a complete asymptotically flat L^∞-metric on M^n, which clearly satisfies property (1). Moreover, its scalar curvature is given by

$$f^{\frac{4}{n-2}} R_{\hat{g}} = R_g - \frac{4(n-1)}{n-2}\frac{\Delta_g f}{f} = R_g - \frac{4(n-1)}{n-2}\frac{\hat{\mathcal{C}}^{-1}\varepsilon \hat{\mathbf{c}} u}{f} > 0 \tag{2.36}$$

on $M^n \setminus \mathcal{S}$. Lastly, a computation shows that the new mass takes the form

$$\hat{m} = m + 2\hat{\mathcal{C}}^{-1}\varepsilon \cdot \hat{\mathcal{C}} = m + 2\varepsilon. \tag{2.37}$$

□

We next establish a slightly weaker version of Theorem 2.7, which differs from the desired result by its nonnegative (instead of strictly positive) scalar curvature conclusion.

Proposition 2.12. *Assume the setting and notation of Theorem 2.7. Then for each $\varepsilon > 0$ there exists a complete asymptotically flat L^∞-metric $\tilde{g}$ on M^n with the following properties.*

(1) *It holds that $\|\tilde{g} - g\|_{L^\infty(M^n, g_0)} < \varepsilon$.*

(2) *$\tilde{g}$ is smooth on $M^n \setminus \mathcal{S}$ and $R_{\tilde{g}} \ge 0$ on $M^n \setminus \mathcal{S}$.*

(3) *On the end $\mathcal{E}$ the new metric $\tilde{g}$ is harmonically asymptotically flat.*

(4) *If $\tilde{m}$ and m denote the ADM masses of $\tilde{g}$ and g on $\mathcal{E}$, then $|\tilde{m} - m| < \varepsilon$.*

Proof. In the asymptotic end, choose a smooth one-parameter family of radial cut-off functions $0 \le \phi_s \le 1$, $s \gg 1$, such that

$$\phi_s(r) = 1 \quad \text{for } r \le 2s, \qquad \phi_s(r) = 0 \quad \text{for } r \ge 4s, \tag{2.38}$$

and

$$|\nabla^k \phi_s|_\delta \le C_k s^{-k} \qquad \text{on } \{2s \le r \le 4s\}. \tag{2.39}$$

Extend ϕ_s trivially to the rest of M^n. Let g_{mod} be the following harmonically asymptotically flat model metric on $\mathcal{E}$ with the same mass as g:

$$g_{\text{mod}} = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-2}} \delta, \tag{2.40}$$

and define

$$g_s = \phi_s g + (1-\phi_s) g_{\text{mod}}. \tag{2.41}$$

Then $g_s = g$ on $\{r \leq 2s\}$, while $g_s = g_{\text{mod}}$ on $\{r \geq 4s\}$. In particular, g_s is a complete asymptotically flat L^∞-metric with singular set $\mathcal{S}$, and its chosen end is harmonically asymptotically flat with mass m for each s. Moreover $g_s \to g$ uniformly on $M^n \setminus \mathcal{S}$ as $s \to \infty$.

Now choose a second family of radial cut-off functions $0 \leq \psi_s \leq 1$ with

$$\psi_s(r) = 0 \quad \text{for } r \leq s, \qquad \psi_s(r) = 1 \quad \text{for } r \geq 2s, \tag{2.42}$$

and

$$|\nabla^k \psi_s|_\delta \leq C_k s^{-k} \qquad \text{on } \{s \leq r \leq 2s\}. \tag{2.43}$$

Set

$$\mathbf{c}_s = \frac{n-2}{4(n-1)} \psi_s R_{g_s}. \tag{2.44}$$

Since $g_s = g$ on $\{r \leq 2s\}$ and $R_g \geq 0$ there, while $g_s = g_{\text{mod}}$ for $r \geq 4s$, the negative part of $\mathbf{c}_s$ is supported in the transition annulus $\{2s \leq r \leq 4s\}$. Note that in this annulus $|R_{g_s}| \leq C r^{-q-2}$, and hence

$$\|\mathbf{c}_{s-}\|_{L^{n/2}(M^n, g_s)} \leq C \left(\int_{2s \leq r \leq 4s} r^{-\frac{n}{2}(q+2)} r^{n-1}\, dr \right)^{2/n} \leq C s^{-q} \longrightarrow 0. \tag{2.45}$$

Thus, for s sufficiently large, the smallness hypothesis in Theorem 2.10 is satisfied. We may therefore solve

$$\Delta_{g_s} u_s - \mathbf{c}_s u_s = 0, \qquad u_s \to 1 \quad \text{as } r \to \infty. \tag{2.46}$$

Theorem 2.10 gives a solution u_s bounded below by a positive constant in the essential-infimum sense, which is smooth on $M^n \setminus \mathcal{S}$, and with asymptotic expansion

$$u_s = 1 + \frac{\mathcal{C}_s}{r^{n-2}} + O_{2,\alpha}(r^{2-n-q'}), \qquad \mathcal{C}_s = -\frac{1}{(n-2)\omega_{n-1}} \int_{M^n} \mathbf{c}_s u_s \, dV_{g_s}. \tag{2.47}$$

Note that we may take $q' = 1$ and any $\alpha \in (0,1)$ since $\mathbf{c}_s$ vanishes identically for $r \geq 4s$.

We claim that $\mathcal{C}_s \longrightarrow 0$ as $s \to \infty$. Write

$$\int_{M^n} \mathbf{c}_s u_s \, dV_{g_s} = \int_{M^n} \mathbf{c}_s \, dV_{g_s} + \int_{M^n} \mathbf{c}_s (u_s - 1) \, dV_{g_s}, \tag{2.48}$$

and observe that the energy estimate (2.28) in the proof of Theorem 2.10 gives

$$\|u_s - 1\|_{L^{2n/(n-2)}(M^n, g_s)} \leq 2C_* \|\mathbf{c}_s\|_{L^{2n/(n+2)}(M^n, g_s)}, \tag{2.49}$$

while $\|\mathbf{c}_s\|_{L^{2n/(n+2)}(M^n,g_s)} \to 0$ because $|R_{g_s}| \le Cr^{-q-2}$ on the transition region and $q > (n-2)/2$. Hence Hölder's inequality produces

$$\left| \int_{M^n} \mathbf{c}_s(u_s - 1)\, dV_{g_s} \right| \le \|\mathbf{c}_s\|_{L^{2n/(n+2)}} \|u_s - 1\|_{L^{2n/(n-2)}} \longrightarrow 0. \tag{2.50}$$

To see that the first integral on the right-hand side of (2.48) also converges to zero, write $g_s = \delta + h_s$ on the transition annulus $\{2s \le r \le 4s\}$. Note that the asymptotic flatness assumption implies

$$h_s = O(r^{-q}), \qquad \partial h_s = O(r^{-q-1}), \qquad \partial^2 h_s = O(r^{-q-2}), \tag{2.51}$$

uniformly in s. Furthermore, the scalar curvature expansion in Cartesian coordinates has the form

$$R_{g_s} = \partial_i \partial_j (h_s)_{ij} - \partial_i \partial_i (h_s)_{jj} + Q_s, \tag{2.52}$$

where the quadratic error is estimated by

$$|Q_s| \le C\big(|\partial h_s|^2 + |h_s|\, |\partial^2 h_s|\big) \le Cr^{-2q-2}, \tag{2.53}$$

and thus

$$\int_{\{2s \le r \le 4s\}} |Q_s|\, dV_\delta \le Cs^{n-2q-2} \longrightarrow 0. \tag{2.54}$$

It remains to treat the linear part of (2.52). By integration by parts on the annuli and by the ADM flux formula, its integral over the exterior region equals, up to an error tending to zero, the difference between the ADM flux of g_s through a large coordinate sphere and the ADM flux of g through the inner sphere. The outer flux is the ADM mass of g_{mod}, while the inner flux converges to the ADM mass of g. Since g_{mod} was chosen to have the same ADM mass as g, these two fluxes cancel. Consequently

$$\int_{M^n} \psi_s R_{g_s}\, dV_{g_s} = o(1), \tag{2.55}$$

and it follows that $\mathcal{C}_s \to 0$.

Finally define $\tilde{g}_s = u_s^{\frac{4}{n-2}} g_s$ and set $\tilde{g} = \tilde{g}_{s_\varepsilon}$ for sufficiently large s_ε depending on ε. Then $\tilde{g}$ is a complete asymptotically flat L^∞-metric on M^n, which is smooth away from $\mathcal{S}$. Observe that by the energy estimate (2.49) from Theorem 2.10 and Moser iteration $\|u_s - 1\|_{L^\infty(M^n,g_0)} < \varepsilon$ for large s, and thus the same holds for the deviation of $\tilde{g}_s$ from g, establishing property (1). By equation (2.46) and the conformal scalar curvature formula

$$R_{\tilde{g}_s} = u_s^{-\frac{n+2}{n-2}} \left(-\frac{4(n-1)}{n-2} \Delta_{g_s} u_s + R_{g_s} u_s \right) = u_s^{-\frac{4}{n-2}} (1 - \psi_s) R_{g_s}. \tag{2.56}$$

On $\{r \le 2s\}$ one has $\psi_s \le 1$ and $g_s = g$, so $R_{\tilde{g}_s} \ge 0$ in this region away from $\mathcal{S}$. On $\{r \ge 2s\}$ one has $\psi_s = 1$ so $R_{\tilde{g}_s} = 0$ in this region, establishing property (2). Property (3) concerning the harmonic asymptotics of $\tilde{g}_s$, follows directly from the expansion (2.47) and structure of g_{mod}. Furthermore, the relation between the ADM masses is

$$m(\tilde{g}_s) = m(g_s) + 2\mathcal{C}_s = m(g) + 2\mathcal{C}_s. \tag{2.57}$$

Property (4) then follows since $\mathcal{C}_s \to 0$. □

Proof of Theorem 2.7. Consider the case of a single asymptotically flat end. First, apply Proposition 2.12 to obtain a new metric $\tilde{g}$ on M^n satisfying all desired properties, except that its scalar curvature is only known to be nonnegative instead of strictly positive. Next, apply Proposition 2.11 to $(M^n, \tilde{g})$ to obtain $(M^n, \hat{g})$, which satisfies all desired properties including that of positive scalar curvature; note that the conformal change in the proof of this proposition preserves the harmonic asymptotics of the end. Thus, setting $g' = \hat{g}$ yields the desired conclusion. The case of multiple asymptotically flat ends may be treated in an analogous way with straightforward modifications. □

3. Blow-up of Singular Sets

In this section the singular set will be removed by a two-stage blow-up procedure. First, using a Green's function with pole on the codimension greater than $3 - \frac{2}{n}$ singular set, we conformally deform the original L^∞-metric to obtain a complete metric on the regular set, and a positive scalar-curvature-type inequality. We then construct suitable μ-bubbles and derive a stability inequality strong enough to pass to the next dimension. Finally, the GMT singular set arising on the minimizing hypersurface is blown up by a second conformal deformation, allowing the dimension-reduction argument to continue.

3.1. **Blow-up of the singular set for L^∞-metrics.** Let (M^n, g) be a complete harmonically asymptotically Euclidean L^∞-manifold with background metric g_0, and let $\mathcal{S}$ denote its singular set. In order to build the desired conformal factor, we will employ Green's functions with pole at points of $\mathcal{S}$. Choose precompact open neighborhoods U_1 and U_2 of $\mathcal{S}$, such that ∂U_2 is smooth and

$$\mathcal{S} \subset U_1, \qquad \overline{U}_1 \subset U_2. \tag{3.1}$$

The existence of Green's functions for uniformly elliptic operators with L^∞ coefficients, and their basic properties, has been achieved in [23, Theorem 1.1], see also [35] and [11, Lemma A.1]. The relevant form of this result, needed for our purposes, is recorded here.

Lemma 3.1 (Existence and estimates for the Green's function)**.** *Let $(U_2, g|_{U_2})$ and $\mathcal{S}$ be as above. Then there exists a Dirichlet Green's function*

$$G : U_2 \times U_2 \to \mathbb{R} \cup \{\infty\} \tag{3.2}$$

associated with the the Laplace-Beltrami operator on U_2. Moreover, for each $x \in \mathcal{S}$, the function

$$y \mapsto G(x, y) \tag{3.3}$$

is positive and g-harmonic on $U_2 \setminus \mathcal{S}$. In addition, there exists a constant $C > 0$, independent of $x \in \mathcal{S}$, such that for all $x \in \mathcal{S}$ and $y \in U_1 \setminus \mathcal{S}$,

$$C^{-1} d_{g_0}(x, y)^{2-n} \le G(x, y) \le C d_{g_0}(x, y)^{2-n}. \tag{3.4}$$

On the regular set $U_2 \setminus \mathcal{S}$, the function $G(x, \cdot)$ is smooth.

These Green's functions will now be used for the first blow-up, making $M^n \setminus \mathcal{S}$ into a complete manifold with scalar curvature satisfying a positivity property. For the remainder of this section, it will be assumed for convenience that the asymptotically flat manifold (M^n, g) has only one end.

Proposition 3.2 (Blow-up of codimension-three singular set)**.** *Let (M^n, g) be a complete harmonically asymptotically flat L^∞-manifold with end $\mathcal{E}$, and assume that its singular set satisfies $\dim_{\mathcal{M}} \mathcal{S} < n - 3 + \frac{2}{n}$. If the scalar curvature R_g is positive on $M^n \setminus \mathcal{S}$, then there exists a complete harmonically asymptotically flat smooth metric g' on $M^n \setminus \mathcal{S}$ with arbitrary ends, and a constant $\varepsilon_n > 0$ such that the following properties hold.*

(1) The metric g' agrees with g on the end $\mathcal{E}$.

(2) For each $\varepsilon \in (0, \varepsilon_n)$, there exists a positive function $\rho \in C^\infty(M^n \setminus \mathcal{S})$ such that $\rho = 1$ on $\mathcal{E}$ and

$$Q := R_{g'} - 2\Delta_{g'} \log \rho - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla \log \rho|^2_{g'} > 0 \quad \text{on } M^n \setminus \mathcal{S}. \tag{3.5}$$

Proof. Step 1: Construction of a Green's function having pole $\mathcal{S}$. Let $\sigma \in (\dim_{\mathcal{M}} \mathcal{S}, n - 3 + \frac{2}{n})$, and choose a value a satisfying

$$\max\left\{1, \frac{2}{n-2}, \frac{1}{n-2-\sigma}\right\} < a < \frac{n}{n-2}. \tag{3.6}$$

Since $\dim_{\mathcal{M}} \mathcal{S} < \sigma$, there exists ℓ_0 such that, for every integer $\ell \geq \ell_0$, one can find a finite cover $\{B^{g_0}_{r_{\ell,j}}(x_{\ell,j})\}_{j=1}^{N_\ell}$ of $\mathcal{S}$ by g_0-geodesic balls, with $x_{\ell,j} \in \mathcal{S}$, satisfying

$$r_{\ell,j} \leq 2^{-\ell}, \qquad \sum_{j=1}^{N_\ell} r^\sigma_{\ell,j} \leq 1. \tag{3.7}$$

After reindexing ℓ, we may assume this holds for all integers $\ell \geq 1$. Define the atomic measure

$$\nu_{\mathcal{S}} := \sum_{\ell=1}^{\infty} \sum_{j=1}^{N_\ell} r_{\ell,j}^{n-2-a^{-1}} \delta_{x_{\ell,j}}. \tag{3.8}$$

Set $p := n - 2 - a^{-1}$, and observe that since $p - \sigma > 0$ it follows from (3.7) that for any measurable set $X \subset U_2$ we have

$$\begin{aligned} \nu_S(X) &\leq \sum_{\ell=1}^{\infty} \sum_{j=1}^{N_\ell} r^p_{\ell,j} = \sum_{\ell=1}^{\infty} \sum_{j=1}^{N_\ell} r^{p-\sigma}_{\ell,j} r^\sigma_{\ell,j} \\ &\leq \sum_{\ell=1}^{\infty} 2^{-\ell(p-\sigma)} \sum_{j=1}^{N_\ell} r^\sigma_{\ell,j} \leq \sum_{\ell=1}^{\infty} 2^{-\ell(p-\sigma)} < \infty. \end{aligned} \tag{3.9}$$

Thus $\nu_{\mathcal{S}}$ is a finite measure. Moreover, since each atom is centered at a point $x_{\ell,j} \in \mathcal{S}$, and $\mathcal{S}$ is closed, $\nu_{\mathcal{S}}$ is supported on $\mathcal{S}$. Next set

$$G_{\mathcal{S}}(y) := \int_{\mathcal{S}} G(x,y)\, d\nu_{\mathcal{S}}(x) = \sum_{\ell=1}^{\infty} \sum_{j=1}^{N_\ell} r_{\ell,j}^{n-2-a^{-1}} G(x_{\ell,j}, y). \tag{3.10}$$

Since $\nu_{\mathcal{S}}$ is finite and the Green's functions are locally uniformly bounded for y in compact subsets of $U_2 \setminus \mathcal{S}$ by Lemma 3.1, this series converges locally uniformly on $U_2 \setminus \mathcal{S}$. Hence $G_{\mathcal{S}}$ is a positive harmonic function on $U_2 \setminus \mathcal{S}$. Furthermore, by interior elliptic regularity, $G_{\mathcal{S}}$ is smooth on $U_2 \setminus \mathcal{S}$.

We note that $G_{\mathcal{S}}$ has pole set $\mathcal{S}$. Indeed, fix $x \in \mathcal{S}$. For each $\ell \geq 1$, choose j_ℓ such that $x \in B^{g_0}_{r_{\ell,j_\ell}}(x_{\ell,j_\ell})$, and set $s_\ell := r_{\ell,j_\ell}$. Then $s_\ell \leq 2^{-\ell}$, and hence $s_\ell \to 0$. If $y \in U_2 \setminus \mathcal{S}$ satisfies $d_{g_0}(y,x) < s_\ell$, then

$$d_{g_0}(y, x_{\ell,j_\ell}) \leq d_{g_0}(y,x) + d_{g_0}(x, x_{\ell,j_\ell}) < 2s_\ell. \tag{3.11}$$

Using the lower Green's function estimate (3.4) and the definition of $\nu_{\mathcal{S}}$, we obtain

$$\begin{aligned} G_{\mathcal{S}}(y) \geq s_\ell^{n-2-a^{-1}} G(x_{\ell,j_\ell}, y) &\geq C^{-1} s_\ell^{n-2-a^{-1}} d_{g_0}(x_{\ell,j_\ell}, y)^{2-n} \\ &\geq C^{-1} 2^{2-n} s_\ell^{n-2-a^{-1}} s_\ell^{2-n} = c\, s_\ell^{-a^{-1}}. \end{aligned} \tag{3.12}$$

It follows that

$$G_{\mathcal{S}}(y) \to +\infty \qquad \text{as } y \to x, \quad y \in U_2 \setminus \mathcal{S}. \tag{3.13}$$

For the completeness of the conformally blown-up metric, however, this pointwise blow-up is not sufficient by itself. What is needed is the stronger integral statement that $G_{\mathcal{S}}^a$ is non-integrable along every curve approaching $\mathcal{S}$. The Wolff-type potential introduced below encodes the scale-by-scale lower bounds for the Green's potential and will be used to prove precisely this nonintegrability.

Consider the Wolff-type potential on U_2,

$$W(x) := \int_0^1 \left(\nu_{\mathcal{S}}(B_r^{g_0}(x)) r^{2-n}\right)^a \, dr. \tag{3.14}$$

Fix $x \in \mathcal{S}$, and as above for each $\ell \geq 1$ choose j_ℓ such that $x \in B^{g_0}_{r_{\ell,j_\ell}}(x_{\ell,j_\ell})$. Note that for every $r \in [s_\ell, 2s_\ell]$, one has $x_{\ell,j_\ell} \in B_r^{g_0}(x)$. Therefore

$$\nu_{\mathcal{S}}(B_r^{g_0}(x)) \geq s_\ell^{n-2-a^{-1}}. \tag{3.15}$$

It follows that, for $r \in [s_\ell, 2s_\ell]$,

$$\left(\nu_{\mathcal{S}}(B_r^{g_0}(x)) r^{2-n}\right)^a \geq c \left(s_\ell^{n-2-a^{-1}} s_\ell^{2-n}\right)^a = c\, s_\ell^{-1}. \tag{3.16}$$

Since $s_\ell \to 0$, we may choose a subsequence ℓ_k such that $2s_{\ell_{k+1}} < s_{\ell_k}$, then the intervals $[s_{\ell_k}, 2s_{\ell_k}]$ are pairwise disjoint. Consequently,

$$W(x) \geq \sum_{k=1}^{\infty} \int_{s_{\ell_k}}^{2s_{\ell_k}} \left(\nu_{\mathcal{S}}(B_r^{g_0}(x)) r^{2-n}\right)^a \, dr \geq c \sum_{k=1}^{\infty} \int_{s_{\ell_k}}^{2s_{\ell_k}} s_{\ell_k}^{-1}\, dr = +\infty. \tag{3.17}$$

Thus W takes the value $+\infty$ at all points of $\mathcal{S}$.

Step 2: Construction of a complete metric on $M^n \backslash \mathcal{S}$. Let φ be a smooth function on M^n satisfying

$$0 \leq \varphi \leq 1, \qquad \varphi = 1 \text{ on } U_1, \qquad \overline{\operatorname{supp} \varphi} \subset U_2. \tag{3.18}$$

For a constant $\tau > 0$, to be chosen later, define

$$w := 1 + \tau \varphi G_{\mathcal{S}}, \qquad g' := w^{2a} g, \tag{3.19}$$

where $\varphi G_{\mathcal{S}}$ is extended by zero outside U_2. Since $\varphi = 0$ on the asymptotically flat end $\mathcal{E}$, we have $g' = g$ on $\mathcal{E}$. Thus, the harmonically asymptotically flat asymptotics are preserved.

We now prove completeness near $\mathcal{S}$. Let $\gamma : [0, L) \to U_1 \setminus \mathcal{S}$ be a g-unit-speed curve such that $\lim_{t \to L} \gamma(t) = x \in \mathcal{S}$. Since g and g_0 are uniformly equivalent on $\overline{U}_2$, there is a constant $C_0 > 0$ such that, for $t \in [L - \eta, L)$ with $\eta > 0$ sufficiently small, we have $d_{g_0}(\gamma(t), x) \leq C_0(L - t)$. Set $r_t := 2C_0(L - t)$. If $z \in B^{g_0}_{r_t}(x)$, then

$$d_{g_0}(z, \gamma(t)) \leq d_{g_0}(z, x) + d_{g_0}(x, \gamma(t)) \leq 2r_t. \tag{3.20}$$

Using the lower Green's function estimate (3.4), and the fact that $\nu_{\mathcal{S}}$ is supported on $\mathcal{S}$, we obtain

$$\begin{aligned} G_{\mathcal{S}}(\gamma(t)) &= \int_{\mathcal{S}} G(z, \gamma(t)) \, d\nu_{\mathcal{S}}(z) \\ &\geq \int_{B^{g_0}_{r_t}(x)} G(z, \gamma(t)) \, d\nu_{\mathcal{S}}(z) \geq c \, \nu_{\mathcal{S}}(B^{g_0}_{r_t}(x)) \, r_t^{2-n}. \end{aligned} \tag{3.21}$$

Therefore, as in [3, Lemma 2.20],

$$\begin{aligned} \int_0^L G_{\mathcal{S}}(\gamma(t))^a \, dt &\geq c \int_{L-\eta}^L \left(\nu_{\mathcal{S}}(B^{g_0}_{r_t}(x)) r_t^{2-n} \right)^a \, dt \\ &= c(2C_0)^{-1} \int_0^{2C_0 \eta} \left(\nu_{\mathcal{S}}(B^{g_0}_r(x)) r^{2-n} \right)^a \, dr = +\infty, \end{aligned} \tag{3.22}$$

where the last equality follows from $W(x) = +\infty$.

Since $\varphi = 1$ on U_1, we have $w \geq \tau G_{\mathcal{S}}$ along γ for t sufficiently close to L. Hence

$$L_{g'}(\gamma) = \int_0^L w(\gamma(t))^a \, dt \geq \tau^a \int_{L-\eta}^L G_{\mathcal{S}}(\gamma(t))^a \, dt = +\infty.$$

Thus every curve approaching $\mathcal{S}$ has infinite g'-length. Since $g' = g$ on the asymptotically flat end and g' is smooth on $M^n \setminus \mathcal{S}$, it follows that g' is complete on $M^n \setminus \mathcal{S}$.

Step 3: The positivity of Q. Fix $\varepsilon > 0$, to be chosen sufficiently small, and set

$$\mu := \frac{n-1}{n} + \varepsilon, \qquad \rho := w^b, \tag{3.23}$$

where $b \in \mathbb{R}$ will be chosen below. Since $g' = w^{2a} g$, the conformal transformation formulas give

$$\begin{aligned} w^{2a} R_{g'} &= R_g - 2a(n-1)\Delta_g \log w - (n-1)(n-2)a^2 |\nabla \log w|_g^2, \\ w^{2a} \Delta_{g'} \log w &= \Delta_g \log w + a(n-2)|\nabla \log w|_g^2, \\ w^{2a} |\nabla \log w|_{g'}^2 &= |\nabla \log w|_g^2. \end{aligned} \tag{3.24}$$

Using $\log \rho = b \log w$, we obtain

$$\begin{aligned} w^{2a} Q = R_g &- 2\big(b + a(n-1)\big)\Delta_g \log w \\ &- \big(2ab(n-2) + (n-1)(n-2)a^2 + \mu b^2\big) |\nabla \log w|_g^2. \end{aligned} \tag{3.25}$$

Since

$$\Delta_g \log w = \frac{\Delta_g w}{w} - |\nabla \log w|_g^2, \tag{3.26}$$

this becomes

$$w^{2a} Q = R_g - 2\big(b + a(n-1)\big)\frac{\Delta_g w}{w} + c_* |\nabla \log w|_g^2, \tag{3.27}$$

where

$$c_* = -\big(2a(n-2) - 2\big)b - \big((n-1)(n-2)a^2 - 2a(n-1)\big) - \mu b^2. \tag{3.28}$$

We choose b so that the coefficient of the gradient term in (3.27) vanishes, namely $c_* = 0$. Because $a > \frac{2}{n-2}$ this quadratic equation has a real root provided

$$\mu < \frac{(2a(n-2) - 2)^2}{4\big((n-1)(n-2)a^2 - 2a(n-1)\big)}. \tag{3.29}$$

Moreover, the choice $a < \frac{n}{n-2}$ implies

$$\frac{(2a(n-2) - 2)^2}{4\big((n-1)(n-2)a^2 - 2a(n-1)\big)} > \frac{n-1}{n}. \tag{3.30}$$

Hence there exists $\varepsilon_n > 0$ such that (3.29) holds whenever

$$0 < \varepsilon < \varepsilon_n, \qquad \mu = \frac{n-1}{n} + \varepsilon. \tag{3.31}$$

For each such ε, choose a real root $b = b(\varepsilon)$ of (3.28). The branch may be chosen so that $b(\varepsilon)$ remains uniformly bounded for $\varepsilon \in (0, \varepsilon_n)$.

With this choice of b, the gradient term in (3.27) vanishes, and it follows that

$$w^{2a} Q = R_g - 2\big(b + a(n-1)\big)\frac{\tau \Delta_g(\varphi G_{\mathcal{S}})}{w}. \tag{3.32}$$

Because $w \geq 1$, and because $b = b(\varepsilon)$ is uniformly bounded for $\varepsilon \in (0, \varepsilon_n)$, there exists a constant $C_1 > 0$, independent of ε, such that

$$w^{2a} Q \geq R_g - C_1 \tau |\Delta_g(\varphi G_{\mathcal{S}})|. \tag{3.33}$$

Observe that $\Delta_g(\varphi G_{\mathcal{S}}) = 0$ on $U_1 \setminus \mathcal{S}$, since $\varphi = 1$ there and $G_{\mathcal{S}}$ is g-harmonic. It also vanishes on $M^n \setminus U_2$, since $\varphi G_{\mathcal{S}}$ is supported in U_2. On the compact set $\overline{U_2 \setminus U_1}$, define

$$C_2 := \sup_{U_2 \setminus U_1} |\Delta_g(\varphi G_{\mathcal{S}})| < \infty, \qquad C_3 := \inf_{U_2 \setminus U_1} R_g > 0. \tag{3.34}$$

Choose $\tau > 0$ sufficiently small so that $C_1 C_2 \tau < C_3$. Then (3.33) gives

$$w^{2a} Q > 0 \qquad \text{on } U_2 \setminus U_1. \tag{3.35}$$

On $U_1 \setminus \mathcal{S}$ and on $M^n \setminus U_2$, we have $\Delta_g(\varphi G_{\mathcal{S}}) = 0$, and hence on those sets $w^{2a} Q = R_g > 0$. Therefore, $Q > 0$ on $M^n \setminus \mathcal{S}$. ☐

3.2. **μ-bubble existence and stability.** Although Proposition 3.2 shows that $M^n \setminus \mathcal{S}$ carries a complete smooth asymptotically flat metric satisfying the scalar-curvature inequality (3.5), this is not by itself sufficient to carry out the dimension-reduction argument in all dimensions. The obstruction is that the coefficient of the gradient term in (3.5) is not preserved under dimension reduction. In this subsection we construct suitable μ-bubbles and derive a stability inequality in which the coefficient of the gradient term is improved enough to allow the dimension-reduction argument to be iterated.

3.2.1. *Potential function.* The metric produced in Proposition 3.2 is complete and harmonically asymptotically flat, but it may have additional ends. In order to construct the required μ-bubbles, we first construct a barrier potential which vanishes on the prescribed asymptotically flat end and tends to $-\infty$ along the auxiliary ends.

Lemma 3.3 (Existence of a barrier potential)**.** *Let (M^n, g) be a complete Riemannian manifold with a harmonically asymptotically flat end $\mathcal{E}$, possibly with additional arbitrary ends, and let $Q \in C^\infty(M^n)$ be positive. There exists a constant $\delta_0 > 0$ such that, for every $\delta \in (0, \delta_0)$, there is a connected open set $\Omega_\delta \subset M^n$ that contains the prescribed end $\mathcal{E}$ with $\Omega_\delta \setminus \mathcal{E}$ precompact, and a smooth function $\Phi \in C^\infty(\Omega_\delta)$ satisfying*

$$\Phi = 0 \quad \text{on } \mathcal{E}, \qquad \Phi(x) \to -\infty \quad \text{as } x \to \partial\Omega_\delta, \tag{3.36}$$

and

$$Q + 2\delta^2 \Phi^2 - 4|\nabla \Phi|_g \geq 0 \quad \text{on } \Omega_\delta. \tag{3.37}$$

Proof. Choose $U \subset M^n$, a connected open neighborhood of the prescribed end $\mathcal{E}$, with smooth compact boundary. Since M^n is complete, the distance function $d_g(\cdot, \partial U)$ is a proper 1-Lipschitz function on $M^n \setminus U$. After smoothing this function and modifying it in a collar of ∂U, we may choose a smooth proper function $\phi : M^n \setminus U \to [0, \infty)$ such that $\phi = 0$ on ∂U and $|\nabla \phi|_g \leq 1$. Choose a regular value $d_0 > 0$ of ϕ, and set

$$U' := U \cup \{x \in M^n \setminus U : \phi(x) < d_0\}. \tag{3.38}$$

Then U' has smooth boundary, $\overline{U} \subset U'$, and $V := U' \setminus \overline{U}$ has compact closure. Moreover, with

$$\partial_0 V := \partial U, \qquad \partial_1 V := \partial U' = \{\phi = d_0\}, \tag{3.39}$$

we have

$$\phi = 0 \quad \text{on } \partial_0 V, \qquad \phi = d_0 \quad \text{on } \partial_1 V, \qquad |\nabla\phi|_g \leq 1. \tag{3.40}$$

Since $Q > 0$ and $\overline{V}$ is compact, there exists $\delta_0 > 0$ appropriately small such that

$$Q > 8\delta_0^2 \quad \text{on } V, \qquad d_0\delta_0^2 < \frac{\pi}{2}. \tag{3.41}$$

For $\delta \in (0, \delta_0)$, set

$$L_\delta := d_0 + \delta^{-2}\cot(\delta^2 d_0), \tag{3.42}$$

and consider the open set

$$\tilde{\Omega}_\delta := U \cup \{x \in M^n \setminus U : \phi(x) < L_\delta\}. \tag{3.43}$$

We define Ω_δ to be the connected component of $\tilde{\Omega}_\delta$ containing the prescribed end $\mathcal{E}$, and note that $\partial\Omega_\delta$ is a subset of the level set $\{\phi = L_\delta\}$. Next, define a barrier function Φ on Ω_δ by

$$\Phi := \begin{cases} 0, & \text{on } U, \\ -2\tan(\delta^2\phi), & \text{on } V, \\ -\dfrac{2}{\cot(\delta^2 d_0) - \delta^2(\phi - d_0)}, & \text{on } \Omega_\delta \setminus U'. \end{cases} \tag{3.44}$$

The two nonzero pieces agree in value along $\partial_1 V$, and both agree with 0 along $\partial_0 V$. After smoothing in arbitrarily small collars of $\partial_0 V$ and $\partial_1 V$, and after decreasing δ_0 if necessary, we obtain a smooth function, still denoted Φ, satisfying the properties below. By (3.42), the denominator in the last line of (3.44) tends to 0^+ as $\phi \to L_\delta$. Hence $\Phi(x) \to -\infty$ as $x \to \partial\Omega_\delta$.

It remains to verify (3.37). On U, we have $\Phi = 0$, and hence the desired inequality holds on this set due to the positivity of Q. On V, using $|\nabla\phi|_g \leq 1$, we obtain

$$|\nabla\Phi|_g \leq 2\delta^2\sec^2(\delta^2\phi) = 2\delta^2\left(1 + \frac{\Phi^2}{4}\right). \tag{3.45}$$

Therefore, on V,

$$Q + 2\delta^2\Phi^2 - 4|\nabla\Phi|_g \geq Q + 2\delta^2\Phi^2 - 8\delta^2\left(1 + \frac{\Phi^2}{4}\right) = Q - 8\delta^2 > 0, \tag{3.46}$$

where the last inequality follows from (3.41). On $\Omega_\delta \setminus U'$, a direct computation gives $|\nabla\Phi|_g \leq \frac{\delta^2}{2}\Phi^2$, and consequently on this region

$$Q + 2\delta^2\Phi^2 - 4|\nabla\Phi|_g \geq Q > 0. \tag{3.47}$$

The smoothing near $\partial_0 V$ and $\partial_1 V$ is performed inside a fixed compact subset of $\overline{V}$, where Q has a positive lower bound; by choosing the smoothing collars sufficiently small and then decreasing δ_0 if necessary, the same desired inequality remains valid there. $\square$

3.2.2. *Existence of μ-bubbles.* Let (M^n, g) be a complete Riemannian manifold with a harmonically asymptotically flat end $\mathcal{E}$ having negative mass $m(\mathcal{E}, g) < 0$, and possibly with additional arbitrary ends. We seek a complete asymptotically flat hypersurface (the μ-bubble), that plays a role analogous to the stable minimal surfaces in classical Schoen-Yau dimensional reduction, but now in the context of arbitrary ends. The existence of such surfaces, which were originally introduced by Gromov [22, Section 9], is well-known [12, 33, 54], and thus here we only outline the argument with some detail pertinent to the current setting.

Let $\rho \in C^\infty(M^n)$ be a positive weight function with $\rho = 1$ on $\mathcal{E}$, and let Φ and Ω_δ be the barrier potential and domain constructed in Lemma 3.3 with respect to a positive $Q \in C^\infty(M^n)$, for some $\delta \in (0, \delta_0)$. Choose asymptotic Cartesian coordinates $x = (x', x^n)$ on the prescribed end $\mathcal{E}$, where $x' \in \mathbb{R}^{n-1}$. For $r, \sigma > 0$ large, define

$$\begin{aligned} \mathcal{Z}_{r,\sigma} &:= \big(\Omega_\delta \setminus \mathcal{E}\big) \cup \{x \in \mathcal{E} : |x'| \le r,\ -\sigma \le x^n \le \sigma\}, \\ \mathcal{P}_r^{\pm\sigma} &:= \{x \in \mathcal{E} : |x'| \le r,\ x^n = \pm\sigma\}, \\ \mathcal{D}_{r,\sigma} &:= \{x \in \mathcal{E} : |x'| = r,\ -\sigma \le x^n \le \sigma\}. \end{aligned} \tag{3.48}$$

Since $\rho = 1$ and $\Phi = 0$ on $\mathcal{E}$, and $m(M^n, g, \mathcal{E}) < 0$, the standard asymptotic barrier estimates imply that, for all r and σ sufficiently large,

$$H + \nu(\log \rho) - \Phi > 0 \qquad \text{at } \mathcal{P}_r^{\pm\sigma} \cup \mathcal{D}_{r,\sigma}, \tag{3.49}$$

where ν is the unit outer normal and H is the mean curvature of these boundary components of $\mathcal{Z}_{r,\sigma}$.

Step 1: The truncated μ-bubble problem with free boundary. Decompose the boundary of $\mathcal{Z}_{r,\sigma}$ as

$$\partial_+ := \mathcal{P}_r^{+\sigma}, \qquad \partial_- := \mathcal{P}_r^{-\sigma} \cup \partial\Omega_\delta, \qquad \partial_0 := \mathcal{D}_{r,\sigma}. \tag{3.50}$$

Let $\Omega_0 \subset \mathcal{Z}_{R,\sigma}$ be a fixed Caccioppoli set such that $\partial_+ \subset \Omega_0$ and $\partial_- \cap \Omega_0 = \emptyset$. We minimize the μ-bubble functional with free boundary

$$\mathcal{F}_{r,\sigma}(\Omega) := \int_{\partial^*\Omega} \rho \, d\mathcal{H}^{n-1} - \int_{\mathcal{Z}_{r,\sigma}} (\chi_\Omega - \chi_{\Omega_0}) \rho \Phi \, d\mathcal{H}^n \tag{3.51}$$

among Caccioppoli sets $\Omega \subset \mathcal{Z}_{r,\sigma}$ satisfying

$$\Omega \Delta \Omega_0 \Subset \mathring{\mathcal{Z}}_{r,\sigma} \cup \partial_0. \tag{3.52}$$

Here χ_Ω is the characteristic function of Ω, $d\mathcal{H}^n$ is the n-dimensional Hausdorff measure, and $\partial^*\Omega$ denotes the reduced boundary relative to the region in which variations are allowed; in particular, the fixed boundary faces ∂_+ and ∂_- are not included in the first variation.

The inequalities (3.49), together with the fact that $\Phi \to -\infty$ as $x \to \partial\Omega_\delta$, provide the required barriers at ∂_+ and ∂_-. Therefore the standard compactness and barrier argument for μ-bubbles [12, Sections 3 & 4] gives a minimizer $\Omega_{r,\sigma}$

of $\mathcal{F}_{r,\sigma}$. The desired free boundary μ-bubble is defined to be the closure of the reduced boundary

$$\Sigma_{r,\sigma} := \overline{\partial^* \Omega}_{r,\sigma}. \tag{3.53}$$

The side-barrier inequality along ∂_0 implies that $\Sigma_{r,\sigma}$ meets ∂_0 orthogonally.

Step 2: Passing to the limit and asymptotics of the end. We now let $r \to \infty$, keeping σ fixed. By the compactness argument for μ-bubbles [33, Theorem 4.5] and almost-minimizing boundaries [19, Lemma A.3], after passing to a subsequence $r_j \to \infty$, the hypersurfaces $\Sigma_{r_j,\sigma}$ converge in the sense of currents to a limiting $(n-1)$-integral current, whose support we denote by Σ^{n-1}; we will refer to this surface as a *μ-bubble associated with* (ρ, Φ). The regularity theory for almost-minimizing hypersurfaces with prescribed mean curvature [19, Theorem A.1] implies that Σ^{n-1} is smooth outside a closed singular set $\mathcal{S}_\Sigma$ of Hausdorff codimension seven or greater. Away from this singular set, the first variation formula implies that

$$H_\Sigma = \Phi - \nu(\log \rho), \tag{3.54}$$

where H_Σ denotes mean curvature with respect to the unit outer normal ν; here the 'outer' direction is determined by the approximating Caccioppoli sets. Furthermore, the corresponding Minkowski dimension bound for the singular set of μ-bubbles follows from the Cheeger–Naber [10, Theorem 5.8]/Naber–Valtorta [42, Theorem 1.6] stratification argument, as adapted to μ-bubbles by Brendle–Wang [6, Theorem 3.34], thus

$$\dim_{\mathcal{M}}(\mathcal{S}_\Sigma) \leq n - 8. \tag{3.55}$$

Moreover, $\mathcal{S}_\Sigma$ is contained in a compact subset of M^n. In fact, stronger statements about the asymptotics of the limiting μ-bubble Σ^{n-1} are valid. Indeed, in [33, Theorem 4.5] asymptotic estimates combined with Allard's regularity theorem show that there exists $r_0 > 0$ large enough, so that $\Sigma^{n-1} \cap \{|x'| > r_0\}$ may be expressed as a graph $x^n = z(x')$ for a smooth function z satisfying

$$\begin{aligned} z - a_0 &\in C^{3,\alpha}_{-\epsilon}(\mathbb{R}^{n-1} \setminus B_{r_0}) \qquad \text{if } n = 3, \\ z - \left(a_0 + b_0 |x'|^{3-n}\right) &\in C^{3,\alpha}_{3-n-\epsilon}(\mathbb{R}^{n-1} \setminus B_{r_0}) \qquad \text{if } n \geq 4, \end{aligned} \tag{3.56}$$

for some $a_0, b_0 \in \mathbb{R}$, $\epsilon > 0$, and $\alpha \in (0, 1)$. Since the ambient end $(\mathcal{E}, g)$ is harmonically asymptotically flat, it follows that the corresponding asymptotic end of the μ-bubble $(\mathcal{E}_\Sigma, g_\Sigma)$ is also harmonically asymptotically flat, and has zero mass $m(\mathcal{E}_\Sigma, g_\Sigma) = 0$ if $n \geq 4$. We now record these results.

Proposition 3.4 (μ-bubble existence). *Consider a complete Riemannian manifold (M^n, g) with a harmonically asymptotically flat end $\mathcal{E}$ having negative mass $m(\mathcal{E}, g) < 0$, and possibly with additional arbitrary ends. Let $\rho \in C^\infty(M^n)$ be a positive weight function with $\rho = 1$ on $\mathcal{E}$, and let Φ and Ω_δ be the barrier potential and domain constructed in Lemma 3.3 with respect to a positive $Q \in C^\infty(M^n)$, for some $\delta \in (0, \delta_0)$. Then there exists a μ-bubble $\Sigma^{n-1} \subset \Omega_\delta$ associated with (ρ, Φ), which is smooth outside a closed singular set $\mathcal{S}_\Sigma$ residing within a compact* *subset of M^n, such that if it is nonempty*

$$\dim_{\mathcal{M}} \mathcal{S}_\Sigma \leq n-8. \tag{3.57}$$

Moreover, (Σ^{n-1}, g_Σ) satisfies equation (3.54) away from the singular set, and has a single harmonically asymptotically flat end[2] *$\mathcal{E}_\Sigma$, with the property that if $n \geq 4$ then its mass vanishes, $m(\mathcal{E}_\Sigma, g_\Sigma) = 0$.*

Remark 3.5. *Note that since Σ^{n-1} is contained in Ω_δ and $\Omega_\delta \setminus \mathcal{E}$ is a precompact subset of M^n, the functions ρ and $|\nabla\rho|_g$ are uniformly controlled, namely there is a positive constant C_0 such that*

$$0 < C_0^{-1} \leq \rho \leq C_0, \qquad |\nabla \log \rho|_g \leq C_0, \qquad \textit{on } \Sigma^{n-1}. \tag{3.58}$$

Moreover, since Σ^{n-1} stays a positive distance away from $\partial\Omega_\delta$, the function $|\Phi|$ is uniformly bounded on the μ-bubble. Thus, equation (3.54) implies that

$$|H_\Sigma| \leq C_1 \qquad \textit{on } \Sigma^{n-1} \setminus \mathcal{S}_\Sigma, \tag{3.59}$$

for some positive constant C_1.

3.2.3. *The stability inequality.* In this subsection we derive the stability inequality satisfied by the limiting μ-bubble. The stability obtained here is stronger than the usual compactly supported stability: following the strong-stability argument of Lesourd–Unger–Yau [33], with the height-picking step replaced by the free-boundary approximation, one may pass to the limit in stability inequalities for variations which do not vanish at infinity. This yields a strong stability inequality for test functions which differ from a constant by a weighted Sobolev function. We then combine this strong stability with the scalar-curvature positivity property of the ambient metric to obtain a refined inequality as in [6, Definition 1.3] although with a modified coefficient, suitable for the dimension-reduction argument.

Proposition 3.6 (Stability inequality)**.** *Consider the hypotheses and setting of Proposition 3.4, and let $\Sigma^{n-1} \subset \Omega_\delta$ be the constructed μ-bubble associated with (ρ, Φ). For any $c \in \mathbb{R}$ let $f \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma)$ be a test function such that f vanishes in a neighborhood of $\mathcal{S}_\Sigma$, and $f - c \in W^{1,2}_{-\frac{n-3}{2}}(\mathcal{E}_\Sigma)$ when $n \geq 4$ or $\operatorname{supp}(f-c)$ is compact in $\overline{\mathcal{E}}_\Sigma$ when $n = 3$. Then*

$$\int_{\Sigma^{n-1}} \left(\rho|\nabla_\Sigma f|^2_{g_\Sigma} - \rho\left(Ric(\nu,\nu) + |A_\Sigma|^2 - \nabla^2 \log\rho(\nu,\nu) + \nu(\Phi)\right) f^2\right) dV_{g_\Sigma} \geq 0. \tag{3.60}$$

Furthermore, if additionally

$$Q = R_g - 2\Delta_g \log\rho - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla \log \rho|^2_g > 0 \qquad \textit{on } \Omega_\delta \tag{3.61}$$

[2]When $n = 3$, the 2-dimensional end $\mathcal{E}_\Sigma$ is still referred to as harmonically asymptotically flat, where Definition 2.4 is extended to accommodate this case by removing the mass term from (2.4) as in Remark 2.6. In this situation, the mass is defined to be zero.

for some $\varepsilon > 0$, then there exists a constant $\delta_{n,\varepsilon} < \delta_0$ depending only on n and ε, and a positive function $Q_1 \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma)$, such that for any $\delta \in (0, \delta_{n,\varepsilon})$ it holds that

$$\begin{aligned}&\int_{\Sigma^{n-1}}\left(\rho|\nabla_\Sigma f|^2_{g_\Sigma}+\frac{1}{2}\rho\left(R_{g_\Sigma}-2\Delta_{g_\Sigma}\log\rho-\left(\frac{n-1}{n}+\varepsilon\right)|\nabla_\Sigma\log\rho|^2_{g_\Sigma}\right)f^2\right)dV_{g_\Sigma}\\ \geq&\int_{\Sigma^{n-1}}\rho Q_1 f^2 dV_{g_\Sigma}.\end{aligned} \tag{3.62}$$

Proof. The inequality (3.60) follows from the second variation formula for μ-bubbles with free boundaries. Indeed, ordinary stability yields the inequality for compactly supported variations. In the present free-boundary construction, the approximating bubbles meet the side cylinder $\partial_0 = \mathcal{D}_{r,\sigma}$ freely. Since the vertical vector field ∂_{x^n} is tangent to ∂_0, the free-boundary stability inequality may be applied to variations which do not vanish on ∂_0. This plays the same role as the height-picking argument in [33, Section 4.3]. Passing to the limit as $r \to \infty$ gives the strong stability inequality for the limiting noncompact μ-bubble, namely for test functions f with $f - c$ in the appropriate weighted Sobolev space, as in [33, Theorem 4.21].

It remains to verify the stability inequality (3.62). All calculations to follow will be computed on the regular set of Σ^{n-1}. Recall that two traces of the Gauss equations combined with the Cauchy-Schwarz inequality produces

$$\begin{aligned}Ric_g(\nu,\nu)+|A_\Sigma|^2&=\frac{1}{2}R_g-\frac{1}{2}R_{g_\Sigma}+\frac{1}{2}H_\Sigma^2+\frac{1}{2}|A_\Sigma|^2\\&\geq\frac{1}{2}R_g-\frac{1}{2}R_{g_\Sigma}+\frac{n}{2(n-1)}H_\Sigma^2\end{aligned} \tag{3.63}$$

where A_Σ denotes the second fundamental form, and the standard decomposition of ambient Laplacian with respect to a hypersurface gives

$$\nabla^2\log\rho(\nu,\nu)=\Delta_g\log\rho-\Delta_{g_\Sigma}\log\rho-H_\Sigma\,\nu(\log\rho). \tag{3.64}$$

It follows that

$$\begin{aligned}&-Ric(\nu,\nu)-|A_\Sigma|^2+\nabla^2\log\rho(\nu,\nu)-\nu(\Phi)\\ \leq\;&\frac{1}{2}R_{g_\Sigma}-\Delta_{g_\Sigma}\log\rho-\left(\frac{1}{2}R_g-\Delta_g\log\rho\right)-\frac{n}{2(n-1)}H_\Sigma^2-H_\Sigma\,\nu(\log\rho)+|\nu(\Phi)|\\ \leq\;&\frac{1}{2}R_{g_\Sigma}-\Delta_{g_\Sigma}\log\rho-\frac{1}{2}Q-\frac{1}{2}\left(\frac{n-1}{n}+\varepsilon\right)|\nabla\log\rho|_g^2+|\nu(\Phi)|\\ &-\frac{n}{2(n-1)}\Phi^2+\frac{1}{n-1}\Phi\,\nu(\log\rho)+\frac{n-2}{2(n-1)}|\nu(\log\rho)|^2,\end{aligned} \tag{3.65}$$

where we have used (3.54) and (3.61). The right-hand side of the last inequality can be decomposed into the following three parts.

Part 1. $\frac{1}{2}R_{g_\Sigma} - \Delta_{g_\Sigma}\log\rho - \frac{1}{2}\left(\frac{n-1}{n}+\varepsilon\right)|\nabla_\Sigma\log\rho|^2_{g_\Sigma} - \frac{1}{4}Q$

Part 2. $-\frac{\delta^2}{2}\Phi^2 + |\nu(\Phi)| - \frac{1}{4}Q \leq 0$

Part 3. $-\left(\frac{n}{2(n-1)} - \frac{\delta^2}{2}\right)\Phi^2 + \frac{1}{n-1}\Phi\,\nu(\log\rho) - \left(\frac{1}{2n(n-1)}+\frac{\varepsilon}{2}\right)|\nu(\log\rho)|^2 \leq 0$

Observe that Part 1 is the desired part, while the nonpositivity of Part 2 follows from (3.37) for $\delta\in(0,\delta_0)$. Next, choose a positive $\delta_{n,\varepsilon}<\delta_0$ depending on n and ε so that

$$\left(\frac{1}{n-1}\right)^2 - 4\left(\frac{n}{2(n-1)} - \frac{\delta_{n,\varepsilon}^2}{2}\right)\left(\frac{1}{2n(n-1)}+\frac{\varepsilon}{2}\right) < 0. \tag{3.66}$$

Then for each $\delta\in(0,\delta_{n,\varepsilon})$ the inequality of Part 3 is satisfied. It follows that for such δ we have

$$\begin{aligned}&- Ric(\nu,\nu) - |A_\Sigma|^2 + \nabla^2\log\rho(\nu,\nu) - \nu(\Phi)\\ &\leq \frac{1}{2}R_{g_\Sigma} - \Delta_{g_\Sigma}\log\rho - \frac{1}{2}\left(\frac{n-1}{n}+\varepsilon\right)|\nabla_\Sigma\log\rho|^2_{g_\Sigma} - \frac{1}{4}Q.\end{aligned} \tag{3.67}$$

Inequality (3.62) is now obtained by applying (3.60) and setting $Q_1 = \frac{1}{4}Q$. $\square$

3.3. **The second blow up.** The purpose of this subsection is to find a complete metric on $\Sigma^{n-1}\setminus\mathcal{S}_\Sigma$ by a conformal blow-up of the singular set, while preserving the ADM mass. The next result will place the blown-up μ-bubble in the context of so called weak n-data sets, which are studied in the next section.

Theorem 3.7 (Blow-up of GMT singularities)**.** *Assume the hypotheses and setting of Proposition 3.6, with $Q\in C^\infty(M^n)\cap C^{0,\alpha}_{-n-q_0}(\mathcal{E}_\Sigma)$ where $\alpha\in(0,1)$ and $q_0>0$. Then there exists a positive constant ε_n' such that for any $\varepsilon\in(0,\varepsilon_n')$, there exists a positive function $Q'\in C^\infty(\Sigma^{n-1}\setminus\mathcal{S}_\Sigma)\cap C^{0,\alpha}_{-n-q_0}(\mathcal{E}_\Sigma)$, a positive function $\rho'\in C^\infty(\Sigma^{n-1}\setminus\mathcal{S}_\Sigma)$, and a complete harmonically asymptotically flat metric g_Σ' on $\Sigma^{n-1}\setminus\mathcal{S}_\Sigma$ with arbitrary ends such that*

$$\rho' = 1 \ \textit{on}\ \mathcal{E}_\Sigma, \quad \textit{and} \quad m(\mathcal{E}_\Sigma, g_\Sigma') = 0. \tag{3.68}$$

Furthermore, for any $c\in\mathbb{R}$ and $f\in C^\infty(\Sigma^{n-1}\setminus\mathcal{S}_\Sigma)$ with the property that f vanishes in a neighborhood of $\mathcal{S}_\Sigma$, and $f-c\in W^{1,2}_{-\frac{n-3}{2}}(\mathcal{E}_\Sigma)$ when $n\geq 4$ or $\operatorname{supp}(f-c)$ *is compact in $\overline{\mathcal{E}}_\Sigma$ when $n=3$, it holds that*

$$\begin{aligned}&\int_{\Sigma^{n-1}}\left(\rho'|\nabla_\Sigma f|^2_{g_\Sigma'} + \frac{1}{2}\rho'\left(R_{g_\Sigma'} - 2\Delta_{g_\Sigma'}\log\rho' - \left(\frac{n-1}{n}+\varepsilon\right)|\nabla_\Sigma\log\rho'|^2_{g_\Sigma'}\right)f^2\right)dV_{g_\Sigma'}\\ &\geq \int_{\Sigma^{n-1}}\rho' Q' f^2 dV_{g_\Sigma'}.\end{aligned} \tag{3.69}$$

3.3.1. *Local blow-up function.* Because the geometry of Σ^{n-1} may be poorly controlled near the singular set $\mathcal{S}_\Sigma$, we cannot directly construct a blow-up function on Σ^{n-1} by using a Green's function with pole set $\mathcal{S}_\Sigma$. Instead, we use the ambient manifold (M^n, g). The ambient distance function has controlled Hessian estimates, and this allows us to construct a local blow-up function on Σ^{n-1} which replaces the Green's function used in the first blow-up. We first establish a local blow-up function following the construction of Brendle–Wang [6, Section 3.7]. In order for this to be meaningful we must have $n \geq 8$.

Lemma 3.8 (Blow-up function near singularities). *Let $U_1 \subset M^n$ be a precompact open set containing $\mathcal{S}_\Sigma$, and assume that $n \geq 8$. Then there is a precompact open set $U_2 \subset M^n$ with*

$$\mathcal{S}_\Sigma \subset \overline{U}_2 \subset U_1, \tag{3.70}$$

and a nonnegative function $\psi_{\mathcal{S}} \in C^\infty(M^n \setminus \mathcal{S}_\Sigma)$ with $\operatorname{supp}\psi_{\mathcal{S}} \subset U_1$ such that

$$\psi_{\mathcal{S}}(x) \geq C d_g(x, \mathcal{S}_\Sigma)^{-2} \quad \text{for } x \in U_2 \setminus \mathcal{S}_\Sigma \tag{3.71}$$

and for some constant $C > 0$, and

$$\Delta_{g_\Sigma}\psi_{\mathcal{S}} + (n-3)(n^{-1} - \varepsilon) g_\Sigma(\nabla_\Sigma \psi_{\mathcal{S}}, \nabla_\Sigma \log\rho) < 0 \quad \text{on } (\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap U_2. \tag{3.72}$$

Proof. Step 1: Construction of the one-point blow-up function. Choose a sufficiently small $t_* > 0$ so that $\{x \in M^n : d_g(x, \mathcal{S}_\Sigma) \leq \sqrt{t_*}\}$ is contained in U_1, and pick a smooth monotone nonnegative function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ with

$$\zeta(t) = t^{\frac{3-n}{2}} + t^{\frac{7-2n}{4}} \text{ for } t \in (0, t_*/2], \qquad \zeta(t) = 0 \text{ for } t \in [t_*, \infty). \tag{3.73}$$

For any point $p \in \mathcal{S}_\Sigma$ define

$$\eta_p(x) = d_g^2(x, p), \qquad \psi_p(x) = \zeta(\eta_p(x)). \tag{3.74}$$

Note that by choosing t_* smaller if necessary to achieve $\sqrt{t_*} < \frac{1}{2}\mathrm{inj}_{(M^n,g)}(p)$ for all $p \in \mathcal{S}_\Sigma$, we can ensure that $\psi_p \in C^\infty(M^n \setminus \mathcal{S}_\Sigma)$. Moreover, $\operatorname{supp}\psi_p \subset U_1$. Since $\overline{U}_1$ is compact, there exists a constant $C > 0$ such that on the (M^n, g)-geodesic ball $B_{\sqrt{t_*/2}}(p)$ it holds that

$$|\nabla^2\eta_p - 2g|_g \leq C\eta_p, \quad \psi_p = \eta_p^{\frac{3-n}{2}} + \eta_p^{\frac{7-2n}{4}}, \quad |\nabla\psi_p|_g \leq C(\eta_p^{\frac{2-n}{2}} + \eta_p^{\frac{5-2n}{4}}), \tag{3.75}$$

and

$$\begin{aligned}\nabla^2\psi_p = &\Big(\left(\tfrac{3-n}{2}\right)\eta_p^{\frac{1-n}{2}} + \left(\tfrac{7-2n}{4}\right)\eta_p^{\frac{3-2n}{4}}\Big)\nabla^2\eta_p \\ &+ \Big(\left(\tfrac{3-n}{2}\right)\left(\tfrac{1-n}{2}\right)\eta_p^{-\frac{1+n}{2}} + \left(\tfrac{7-2n}{4}\right)\left(\tfrac{3-2n}{4}\right)\eta_p^{-\frac{1+2n}{4}}\Big) d\eta_p \otimes d\eta_p.\end{aligned} \tag{3.76}$$

Combining this with (3.58) and (3.59), on $(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap B_{\sqrt{t_*/2}}(p)$, produces

$$\begin{aligned}\Delta_{g_\Sigma}\psi_p + &(n-3)(n^{-1} - \varepsilon) g_\Sigma(\nabla_\Sigma\psi_p, \nabla_\Sigma \log\rho) \\ &\leq \Delta_g\psi_p - \nabla^2\psi_p(\nu,\nu) - H_\Sigma\,\nu(\psi_p) + C_1|\nabla\psi_p|_g \\ &\leq \Delta_g\psi_p - \nabla^2\psi_p(\nu,\nu) + C_2|\nabla\psi_p|_g \\ &\leq -\left(\tfrac{2n-7}{4}\right)\eta_p^{\frac{3-2n}{4}} + C_3(\eta_p^{\frac{2-n}{2}} + \eta_p^{\frac{3-n}{2}} + \eta_p^{\frac{5-2n}{4}} + \eta_p^{\frac{7-2n}{4}}) \leq -\left(\tfrac{2n-7}{8}\right)\eta_p^{\frac{3-2n}{4}}\end{aligned} \tag{3.77}$$

where the last two inequalities follow by choosing t_* even smaller, if necessary; here and below uniform positive constants will be denoted by C_i, $i \in \mathbb{Z}_+$. Moreover, on the complimentary domain $(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap B_{\sqrt{t_*/2}}(p)^c$ we have

$$\Delta_{g_\Sigma}\psi_p + (n-3)(n^{-1}-\varepsilon)g_\Sigma(\nabla_\Sigma\psi_p, \nabla_\Sigma \log\rho) \leq C_4. \tag{3.78}$$

Step 2: Construction of the desired blow-up function. A subset $\mathcal{T} \subset \mathcal{S}_\Sigma$ is called an *r-separated subset* if $B_r(p) \cap B_r(p') = \emptyset$ for any two distinct points $p, p' \in \mathcal{T}$. Set $r_l = 2^{-l}\sqrt{t_*}$ and take a maximal r_l-separated subset $\mathcal{T}_l \subset \mathcal{S}_\Sigma$, for each $l \in \mathbb{Z}_+$. The Minkowski dimension bound $\dim_{\mathcal{M}} \mathcal{S}_\Sigma \leq n-8$, and compactness of the singular set, imply that $\mathcal{T}_l$ is a finite and satisfies the cardinality inequality $|\mathcal{T}_l| \leq C_5 r_l^{\mathring{a}-n}$, for any $\mathring{a} < 8$. By restricting further so that $\mathring{a} \in (5,8)$, we also obtain

$$\sum_{l=1}^{\infty} r_l^{n-5}|\mathcal{T}_l| < \infty. \tag{3.79}$$

Define the blow-up function $\psi_{\mathcal{S}} : M^n \setminus \mathcal{S}_\Sigma \to \mathbb{R}_+$ by

$$\psi_{\mathcal{S}}(x) = \sum_{l=1}^{\infty}\sum_{p\in\mathcal{T}_l} r_l^{n-5}\psi_p(x), \tag{3.80}$$

and note that

$$\psi_{\mathcal{S}}(x) \leq C_6 \sum_{l=1}^{\infty} r_l^{n-5}|\mathcal{T}_l| d_g(x,\mathcal{S}_\Sigma)^{3-n} \leq C_7 d_g(x,\mathcal{S}_\Sigma)^{3-n}. \tag{3.81}$$

This shows that $\psi_{\mathcal{S}}$ is well-defined, and similar estimates show that it is smooth on $M^n \setminus \mathcal{S}_\Sigma$. Moreover, $\operatorname{supp}\psi_{\mathcal{S}} \subset U_1$.

Step 3: Construct U_2 and verify the properties of $\psi_{\mathcal{S}}$. Let $t_{**} \in (0, t_*)$ be sufficiently small, to be chosen below, and set

$$U_2 := \{\, x \in M^n : d_g(x,\mathcal{S}_\Sigma) < t_{**}\}.$$

Take $l_* > 1$ depending on t_{**} such that for any $x \in U_2 \setminus \mathcal{S}_\Sigma$, there is an integer $l_x > l_*$ with the property

$$r_{l_x+1} \leq d_g(x,\mathcal{S}_\Sigma) \leq r_{l_x}. \tag{3.82}$$

This ensures the existence of a point $p_{l_x} \in \mathcal{T}_{l_x}$ with $d_g(x, p_{l_x}) \leq 2r_{l_x}$. It follows that

$$\psi_{\mathcal{S}}(x) \geq r_{l_x}^{n-5}\psi_{p_{l_x}}(x) = r_{l_x}^{n-5} d_g(x,p_{l_x})^{3-n} \geq C_8 r_{l_x}^{-2} \geq C_9 d_g(x,\mathcal{S}_\Sigma)^{-2}, \tag{3.83}$$

which establishes (3.71).

It remains to verify (3.72). Let $x \in (\Sigma^{n-1}\setminus\mathcal{S}_\Sigma) \cap U_2$. For convenience set

$$L_\Sigma := \Delta_{g_\Sigma} + (n-3)(n^{-1}-\varepsilon)g_\Sigma(\nabla_\Sigma(\cdot), \nabla_\Sigma\log\rho). \tag{3.84}$$

By the choice of l_*, the point x lies in $B_{\sqrt{t_*/2}}(p_{l_x})$. Hence (3.77) gives

$$L_\Sigma\psi_{p_{l_x}}(x) \leq -C_{10}\, r_{l_x}^{\frac{3-2n}{2}} \tag{3.85}$$

for some constant $C_{10} > 0$, while (3.78) implies that all other terms appearing in the sum $L_\Sigma \psi_{\mathcal{S}}$ have a uniform upper bound $L_\Sigma \psi_p(x) \le C_{11}$. Therefore

$$L_\Sigma \psi_{\mathcal{S}}(x) \le -C_{10}\, r_{l_x}^{-\frac{7}{2}} + C_{11} \sum_{\ell=1}^{\infty} \sum_{p\in\mathcal{T}_\ell} r_\ell^{n-5} \le -C_{10}\, r_{l_x}^{-\frac{7}{2}} + C_{12}. \tag{3.86}$$

Finally, by choosing t_{**} sufficiently small, we can ensure that r_{l_x} is sufficiently small for every $x \in U_2 \setminus \mathcal{S}_\Sigma$, to achieve $-C_{10}\, r_{l_x}^{-\frac{7}{2}} + C_{12} < 0$, which yields the desired result. □

3.3.2. *Gobal blow-up function.*

Lemma 3.9. *Consider the hypotheses and setting of Proposition 3.6, with $n \ge 8$. Then there exists a function $w \in C^\infty(M^n \setminus \mathcal{S}_\Sigma)$ with $w \ge 1$, such that $w = 1$ on $\mathcal{E}_\Sigma$, for some constant $C > 0$ it holds that*

$$w(x) \ge C d_g(x, \mathcal{S}_\Sigma)^{-2} \quad \textit{for } x \in U_2 \setminus \mathcal{S}_\Sigma \tag{3.87}$$

where $U_2 \supset \mathcal{S}_\Sigma$ is as in Lemma 3.8, and

$$Q_1 - \frac{n+1}{n-3}\left(\frac{\Delta_{g_\Sigma} w}{w} + (n-3)(n^{-1}-\varepsilon) g_\Sigma(\nabla_\Sigma \log\rho, \nabla_\Sigma \log w)\right) > 0 \tag{3.88}$$

on $\Sigma^{n-1} \setminus \mathcal{S}_\Sigma$.

Proof. Let $\tau > 0$ be a constant to be determined, and set

$$w = 1 + \tau\psi_{\mathcal{S}}. \tag{3.89}$$

where τ is a positive constant to be determined. Then immediately $w \ge 1$, and also (3.71) implies (3.87). Furthermore, since $\operatorname{supp}\psi_{\mathcal{S}} \subset U_1$ we find that (3.88) holds on $\Sigma^{n-1} \cap (M^n \setminus U_1)$ due to the positivity of Q_1. On the other hand, (3.72) implies (3.88) on $(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap U_2$. Moreover, since $U_1 \setminus U_2$ is precompact and Q_1 is positive there, it follows that we can find $\tau > 0$ to ensure that (3.88) holds on $\Sigma^{n-1} \cap (U_1 \setminus U_2)$ as well. □

3.3.3. *Proof of Theorem 3.7.* Consider a complete Riemannian manifold (M^n, g) with a harmonically asymptotically flat end $\mathcal{E}$ having negative mass $m(\mathcal{E}, g) < 0$, and possibly with additional arbitrary ends. Let $\rho \in C^\infty(M^n)$ be a positive weight function with $\rho = 1$ on $\mathcal{E}$, and let Φ and Ω_δ be the barrier potential and domain constructed in Lemma 3.3 with respect to a positive $Q \in C^\infty(M^n) \cap C^{0,\alpha}_{-n-q_0}(\mathcal{E}_\Sigma)$, for some $\delta \in (0, \delta_0)$. Let $\Sigma^{n-1} \subset \Omega_\delta$ be the μ-bubble associated with (ρ, Φ) given by Proposition 3.4, with singular set denoted by $\mathcal{S}_\Sigma$. For $n \ge 8$ define

$$g'_\Sigma = w^{\frac{n+1}{n-3}} g_\Sigma, \qquad \rho' = w^{-\frac{n+1}{2}}\rho, \tag{3.90}$$

where the positive function $w \in C^\infty(M^n \setminus \mathcal{S}_\Sigma)$ is provided by Lemma 3.9, and for $3 \le n \le 7$ define $g'_\Sigma = g_\Sigma$ and $\rho' = \rho$. Clearly $\rho' \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma)$ and $\rho' = 1$ on $\mathcal{E}_\Sigma$. Moreover, g'_Σ is harmonically asymptotically flat with zero mass, $m(\mathcal{E}_\Sigma, g'_\Sigma) = 0$.

We next show that the metric g'_Σ is complete, and assume that $n \ge 8$ otherwise there is nothing to prove. Since $\Sigma^{n-1} \setminus \mathcal{S}_\Sigma$ has ends, namely $\mathcal{E}_\Sigma$ and $(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap$

U_2, and $(\mathcal{E}_\Sigma, g'_\Sigma)$ is harmonically asymptotically flat, it is sufficient to study the behavior of a g'_Σ-unit-speed curve $\gamma : [0, L) \to (\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap U_2$ with $\lim_{t\to L} \gamma(t) \in \mathcal{S}_\Sigma$. Indeed, observe that (3.87) implies

$$L_{g'_\Sigma}(\gamma) = \int_0^L |\gamma'(t)|_{g_\Sigma} w(\gamma(t))^{\frac{n+1}{2(n-3)}} dt \geq c \int_0^L (L-t)^{-\frac{n+1}{n-3}} dt = \infty \tag{3.91}$$

where $c > 0$ is a constant, from which the desired conclusion follows.

It remains to verify the stability inequality (3.69). Recall the following standard formulas for a conformal change

$$\begin{aligned} w^{\frac{n+1}{n-3}} R_{g'_\Sigma} &= R_{g_\Sigma} - \frac{(n-2)(n+1)}{n-3} \frac{\Delta_{g_\Sigma} w}{w} - \frac{(n+1)(n-2)}{4} |\nabla_\Sigma \log w|^2_{g_\Sigma}, \\ w^{\frac{n+1}{n-3}} \Delta_{g'_\Sigma} \log \rho' &= \Delta_{g_\Sigma} \log \rho' + \frac{n+1}{2} g_\Sigma(\nabla_\Sigma \log w, \nabla_\Sigma \log \rho'). \end{aligned} \tag{3.92}$$

Therefore by setting

$$Q' = \frac{1}{2} w^{-\frac{n+1}{n-3}} Q_1, \tag{3.93}$$

a calculation combined with (3.88) shows that

$$\begin{aligned} &R_{g'_\Sigma} - 2\Delta_{g'_\Sigma} \log \rho' - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla_\Sigma \log \rho'|^2_{g'_\Sigma} - 2Q' \\ &= w^{-\frac{n+1}{n-3}} \left(R_{g_\Sigma} - 2\Delta_{g_\Sigma} \log \rho - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla_\Sigma \log \rho|^2_{g_\Sigma}\right) \\ &- w^{-\frac{n+1}{n-3}} \left(Q_1 + \frac{n+1}{n-3} \left(\frac{\Delta_{g_\Sigma} w}{w} + (n-3)(n^{-1} - \varepsilon) g_\Sigma(\nabla_\Sigma \log \rho, \nabla_\Sigma \log w)\right)\right) \\ &+ \left(\frac{n+1}{4n} - \frac{(n+1)^2}{4}\varepsilon\right) |\nabla_\Sigma \log w|^2_{g_\Sigma} w^{-\frac{n+1}{n-3}} \\ &\geq w^{-\frac{n+1}{n-3}} \left(R_{g_\Sigma} - 2\Delta_{g_\Sigma} \log \rho - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla_\Sigma \log \rho|^2_{g_\Sigma}\right) - 2w^{-\frac{n+1}{n-3}} Q_1, \end{aligned} \tag{3.94}$$

where we have chosen $\varepsilon'_n > 0$ such that $\frac{n+1}{4n} - \frac{(n+1)^2}{4}\varepsilon'_n \geq 0$, and have taken $\varepsilon \in (0, \varepsilon'_n)$. The desired stability inequality now follows by integrating (3.94) against test functions, and applying (3.62). □

4. Weak n-Data Sets and the Generalized PMT

This section isolates the analytic structure needed for the dimension-reduction argument. Brendle–Wang [6, Definition 1.3] introduced n-data sets as the inductive objects in their dimension-descent proof of the positive mass theorem, and our proof of the generalized positive mass theorem follows their overall strategy. The definition used here is a weakened version adapted to the codimension greater than $3 - \frac{2}{n}$ singularities in our main theorem. In particular, the weighted stability inequality below contains a gradient term involving $|\nabla \log \rho|^2$ with a coefficient of smaller absolute value than that appearing in [6].

Definition 4.1. *Let $(M^n, g, \mathcal{E})$ be a complete Riemannian manifold of dimension $n \geq 3$ with arbitrary ends, and having a designated harmonically asymptotically flat end $\mathcal{E}$. A weak n-data set consists of these objects together with a triple (ρ, Q, ε), where ρ, Q are smooth functions on M^n and ε is a constant, all satisfying the following properties.*

(1) The functions ρ and Q are globally positive and $\varepsilon \in (0, \frac{1}{n(n+1)})$.

(2) $Q \in C^{0,\alpha}_{-n-q_0}(\mathcal{E})$ and $\rho - (1 + \beta r^{2-n}) \in C^{2,\alpha}_{2-n-q_0}(\mathcal{E})$ for some $\alpha \in (0, 1)$, $q_0 > 0$, and $\beta \in \mathbb{R}$.

(3) For any $c \in \mathbb{R}$ and $f \in C^\infty(M^n)$ with the property that $(\operatorname{supp} f) \setminus \mathcal{E}$ is precompact, and $f - c \in W^{1,2}_{-\frac{n-2}{2}}(\mathcal{E})$, it holds that

$$\int_{M^n}\left(\rho|\nabla f|_g^2 + \frac{1}{2}\rho\left(R_g - 2\Delta_g \log\rho - \left(\frac{n}{n+1} + \varepsilon\right)|\nabla \log\rho|_g^2\right)f^2\right)dV_g \geq \int_{M^n} \rho Q f^2 dV_g. \tag{4.1}$$

The next theorem is the main result of this section. It may be considered as an extension of the positive mass theorem, and generalizes [6, Theorem 1.5].

Theorem 4.2. *Let $(M^n, g, \mathcal{E}, \rho, Q, \varepsilon)$ be a weak n-data set, then*

$$m(\mathcal{E}, g) + \left(\tfrac{n-2}{n-1}\right)\beta > 0. \tag{4.2}$$

The quantity $m(\mathcal{E}, g) + \left(\frac{n-2}{n-1}\right)\beta$ will be referred to as the *generalized mass* associated with the weak n-data set. There is a loose analogy with the positive mass theorem involving charge [2, Theorem 11.9]. In the present setting, the coefficient β is the monopole coefficient of the weight ρ, and it enters the generalized mass in the same spirit that a charge contributes an additional asymptotic invariant in the charged positive mass theorem, with ρ playing the role of potential for a static electric field. The strict positivity in (4.2) is a consequence of the global positivity of Q. This strict positivity is essential for our proof of Theorem 1.1, which is given at the end of this section. Indeed, Theorem 3.7 produces a weak n-data set with zero generalized mass, and the strict positivity obtained here is precisely that which provides the desired contradiction.

4.1. **Construction of the auxiliary weight factor.** In this subsection we construct a positive auxiliary factor v which modifies the weight ρ. The purpose of this step is to replace ρ by the improved weight $\hat{\rho} := \rho v$ so that the weak stability inequality yields a pointwise scalar-curvature inequality for $\hat{\rho}$. This improved weight will provide the positivity needed for the μ-bubble construction in the next subsection.

Let $(M^n, g, \mathcal{E}, \rho, Q, \varepsilon)$ be a weak n-data set, and define the smooth function

$$\mathcal{Q} := \frac{1}{2}\left(R_g - 2\Delta_g \log\rho - \left(\frac{n}{n+1} + \varepsilon\right)|\nabla \log\rho|_g^2 - Q\right). \tag{4.3}$$

The decay assumptions on ρ, Q, and the harmonically asymptotically flat end imply that

$$\nabla \log \rho \in C^{1,\alpha}_{1-n}(\mathcal{E}), \qquad \mathcal{Q} \in C^{0,\alpha}_{-n-q_1}(\mathcal{E}), \tag{4.4}$$

where $q_1 = \min\{q_0, n-2, \mathring{q}+2-n\} > 0$. Let $\Omega \subset M^n$ be a connected open set that contains $\mathcal{E}$, such that $\Omega \setminus \mathcal{E}$ is precompact. Then from (4.4) and the positivity of ρ, there is a constant $C_0 > 0$ for which

$$C_0^{-1} \leq \rho \leq C_0 \quad \text{and} \quad |\mathcal{Q}| \leq C_0 \quad \text{ on } \Omega. \tag{4.5}$$

In this setting, consider the following Dirichlet problem for $v \in C^\infty(\Omega)$ with prescribed asymptotics at infinity

$$\operatorname{div}_g(\rho\nabla v) - \rho\mathcal{Q}v = 0 \ \text{ in } \Omega, \qquad v = 0 \ \text{ on } \partial\Omega, \qquad v(x) \to 1 \ \text{ as } x \to \infty. \tag{4.6}$$

To study this problem we introduce the bilinear form $\mathcal{B}_\Omega : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{B}_\Omega(u, w) = \int_\Omega (\rho\langle\nabla u, \nabla w\rangle_g + \rho\mathcal{Q}uw)\, dV_g. \tag{4.7}$$

Although the potential term $\mathcal{Q}$ need not be nonnegative, the weak n-data inequality implies that $\mathcal{B}_\Omega$ is coercive in the Sobolev sense.

Lemma 4.3 (Coercive Sobolev inequality)**.** *There exists a constant $C_\Omega > 0$, depending on Ω and the weak n-data, such that for every $u \in W_0^{1,2}(\Omega)$,*

$$\left(\int_\Omega |u|^{\frac{2n}{n-2}}\, dV_g\right)^{\frac{n-2}{n}} \leq C_\Omega \int_\Omega \left(\rho|\nabla u|_g^2 + \rho\mathcal{Q}u^2\right) dV_g. \tag{4.8}$$

Proof. Suppose by way of contradiction that there is a sequence of $\{u_i\}_{i=1}^\infty \subset C_c^\infty(\Omega)$ with

$$\mathcal{B}_\Omega(u_i, u_i) \to 0, \qquad \int_\Omega |u_i|^{\frac{2n}{n-2}} dV_g = 1. \tag{4.9}$$

Then according to (4.1) we have

$$\frac{1}{2}\int_\Omega \rho Q u_i^2 dV_g \leq \mathcal{B}_\Omega(u_i, u_i) \to 0. \tag{4.10}$$

Recall that the Sobolev inequality holds on asymptotically flat manifolds $(\Omega, g|_\Omega)$ with boundary, see [43, Lemma 3.1] for $n = 3$ and note that the same proof holds in all dimensions. Thus, there is a constant $C_* > 0$ such that

$$\left(\int_\Omega |u_i|^{\frac{2n}{n-2}} dV_g\right)^{\frac{n-2}{n}} \leq C_* \int_\Omega |\nabla u_i|_g^2 dV_g. \tag{4.11}$$

Next, observe that $\rho\mathcal{Q} \in L^{\frac{n}{2}}(\Omega)$ by (4.4). Hence, there is a large constant $r_0 > 0$ such that

$$\left(\int_{\mathcal{E}_{r_0}} |\rho\mathcal{Q}|^{\frac{n}{2}} dV_g\right)^{\frac{2}{n}} \leq \frac{1}{2C_*C_0}, \tag{4.12}$$

where in the Cartesian coordinates of the end, $\mathcal{E}_{r_0} := \{x \in \mathcal{E} : |x| \geq r_0\}$. We will decompose Ω into two parts, namely $\mathcal{E}_{r_0}$ and $\Omega \setminus \mathcal{E}_{r_0}$. Since $\Omega \setminus \mathcal{E}_{r_0}$ is precompact and ρQ is positive, there is a constant $C_1 > 0$ such that $\rho Q \geq C_1$ on this domain. It follows from (4.10) that

$$\frac{C_1}{2}\int_{\Omega\setminus\mathcal{E}_{r_0}} u_i^2 dV_g \leq \frac{1}{2}\int_{\Omega} \rho Q u_i^2 dV_g \leq \mathcal{B}_{\Omega}(u_i,u_i) \to 0, \tag{4.13}$$

and with the help of (4.5) this implies

$$\int_{\Omega\setminus\mathcal{E}_{r_0}} |\rho\mathcal{Q}| u_i^2 dV_g \to 0. \tag{4.14}$$

By utilizing again (4.5), as well as (4.11) and (4.12) we obtain

$$\begin{aligned}\mathcal{B}_{\Omega}(u_i,u_i) \geq& \int_{\Omega}\rho|\nabla u_i|_g^2 dV_g + \int_{\mathcal{E}_{r_0}} \rho\mathcal{Q}u_i^2 dV_g - \int_{\Omega\setminus\mathcal{E}_{r_0}} |\rho\mathcal{Q}|u_i^2 dV_g \\ \geq& \frac{1}{C_0}\int_{\Omega}|\nabla u_i|_g^2 dV_g - \|\rho\mathcal{Q}\|_{L^{\frac{n}{2}}(\mathcal{E}_{r_0})}\left(\int_{\Omega}|u_i|^{\frac{2n}{n-2}}dV_g\right)^{\frac{n-2}{n}} - \int_{\Omega\setminus\mathcal{E}_{r_0}}|\rho\mathcal{Q}|u_i^2 dV_g \\ \geq& \frac{1}{2C_*C_0}\left(\int_{\Omega}|u_i|^{\frac{2n}{n-2}}dV_g\right)^{\frac{n-2}{n}} - \int_{\Omega\setminus\mathcal{E}_{r_0}}|\rho\mathcal{Q}|u_i^2 dV_g.\end{aligned} \tag{4.15}$$

Combing this with (4.14) produces

$$\int_{\Omega}|u_i|^{\frac{2n}{n-2}}dV_g \to 0, \tag{4.16}$$

which yields a contradiction with (4.9). □

Proposition 4.4. *Let $(M^n, g, \mathcal{E}, \rho, Q, \varepsilon)$ be a weak n-data set. Then there exists a solution $v \in C^\infty(\overline{\Omega})$ of (4.6), which is positive in Ω, such that*

$$v - \left(1 + \frac{\mathcal{C}}{r^{n-2}}\right) \in C^{2,\alpha}_{2-n-q_1'}(\mathcal{E}), \tag{4.17}$$

where $q_1' = \min\{1, q_1\}$ and

$$(n-2)\omega_{n-1}\mathcal{C} = -\mathcal{B}_{\Omega}(v,v) \leq -\frac{1}{2}\int_{M^n}\rho Q v^2 dV_g < 0. \tag{4.18}$$

Proof. Let $\{\Omega_i\}_{i=1}^{\infty}$ be an exhaustion of Ω by precompact connected open sets whose closure contains $\partial\Omega$, and such that $\partial\Omega_i \setminus \partial\Omega \subset \mathcal{E}$ is a coordinate sphere. For convenience, set

$$L := \Delta_g + \langle \nabla \log\rho, \nabla(\cdot)\rangle_g - \mathcal{Q}. \tag{4.19}$$

Then for each i, the kernel of L in $W_0^{1,2}(\Omega_i)$ is trivial. To see this, observe that if a function $w \in \ker L \cap W_0^{1,2}(\Omega_i)$ then with the help of (4.1) we have

$$0 = -\int_{\Omega_i}\rho w L(w) dV_g = \mathcal{B}_{\Omega_i}(w,w) \geq \frac{1}{2}\int_{\Omega_i}\rho Q w^2 dV_g. \tag{4.20}$$

Hence $w = 0$, since both ρ and Q are positive.

Choose a function $v_0 \in C^\infty(\overline{\Omega})$ such that

$$v_0 = 1 \ \text{ in } \mathcal{E} \qquad \text{and} \qquad v_0 = 0 \ \text{ on } \partial\Omega. \tag{4.21}$$

It follows by elliptic theory that there is then a unique solution $w_i \in C^\infty(\overline{\Omega}_i)$ to the Dirichlet problem

$$L(w_i) = -L(v_0) \text{ in } \Omega_i, \qquad w_i = 0 \text{ on } \partial\Omega_i. \tag{4.22}$$

Multiplying this equation by ρw_i and integrating by parts produces

$$\mathcal{B}_{\Omega_i}(w_i, w_i) = \int_{\Omega_i} \rho w_i L(v_0) dV_g \leq \|\rho L(v_0)\|_{L^{\frac{2n}{n+2}}(\Omega)} \|w_i\|_{L^{\frac{2n}{n-2}}(\Omega_i)}. \tag{4.23}$$

Lemma 4.3 then implies that

$$\|w_i\|_{L^{\frac{2n}{n-2}}(\Omega_i)} \leq C_\Omega \|\rho L(v_0)\|_{L^{\frac{2n}{n+2}}}(\Omega). \tag{4.24}$$

Note that since $v_0 = 1$ in $\mathcal{E}$ we find that $\rho L(v_0) = -\rho\mathcal{Q} \in C^{0,\alpha}_{-n-q_1}(\mathcal{E})$ by (4.4), which implies that the right-hand side of (4.24) is finite.

Using the same convergence argument as in the proof of Theorem 2.10 shows that after passing to a subsequence, $\{w_i\}$ converges to $w \in C^\infty(\overline{\Omega})$ with $w|_{\partial\Omega} = 0$. The decay of the coefficients of L as recorded in (4.4), together with the basic asymptotics analysis of [30, Corollary A.38] implies that $w - \frac{\mathcal{C}}{r^{n-2}} \in C^{2,\alpha}_{2-n-q_1'}(\mathcal{E})$, for some constant $\mathcal{C}$ where $q_1' = \min\{1, q_1\}$. By setting $v = w + v_0$, we find that this function satisfies equations (4.6) and (4.17). Moreover, the monopole estimate (4.18) follows from the stability inequality (4.1), and the positivity of v in Ω arises from a similar argument as in the proof of Theorem 2.10. $\square$

Corollary 4.5. *Let $v \in C^\infty(\overline{\Omega})$ be the function given by Proposition 4.4, and set $\hat{\rho} = \rho v$. If $\varepsilon \in (0, \frac{2n+1}{n(n+1)})$ then it holds that*

$$R_g - 2\Delta_g \log\hat{\rho} - \left(\frac{n-1}{n} + \varepsilon\right)|\nabla \log\hat{\rho}|_g^2 \geq Q \qquad \text{on } \Omega. \tag{4.25}$$

Proof. Using (4.3) and (4.6), and substituting $\log\hat{\rho} = \log\rho + \log v$ we obtain that

$$\begin{aligned} R_g - 2\Delta\log\hat{\rho} &= 2g(\nabla\log\hat{\rho}, \nabla\log v) + \left(\frac{n}{n+1} + \varepsilon\right)|\nabla\log\hat{\rho} - \nabla\log v|_g^2 + Q \\ &= Q + \left(\frac{n}{n+1} + \varepsilon\right)|\nabla\log\hat{\rho}|_g^2 \\ &\quad + 2\left(\frac{1}{n+1} - \varepsilon\right) g(\nabla\log\hat{\rho}, \nabla\log v) + \left(\frac{n}{n+1} + \varepsilon\right)|\nabla\log v|_g^2 \\ &= Q + \left(\frac{n-1}{n} + \varepsilon\right)|\nabla\log\hat{\rho}|_g^2 + \frac{1}{n(n+1)}|\nabla\log\hat{\rho}|_g^2 \\ &\quad + 2\left(\frac{1}{n+1} - \varepsilon\right) g(\nabla\log\hat{\rho}, \nabla\log v) + \left(\frac{n}{n+1} + \varepsilon\right)|\nabla\log v|_g^2. \end{aligned} \tag{4.26}$$

The last three terms form a quadratic expression in $\nabla\log\hat{\rho}$ and $\nabla\log v$. By the allowable range for ε, its discriminant is negative

$$\left(\frac{1}{n+1} - \varepsilon\right)^2 - \frac{1}{n(n+1)}\left(\frac{n}{n+1} + \varepsilon\right) < 0. \tag{4.27}$$

The desired inequality (4.25) now follows. ☐

4.2. **Construction of μ-bubble.** For each $\delta \in (0, \delta_0)$, Lemma 3.3 provides a connected open set Ω_δ and a potential function $\hat{\Phi} \in C^\infty(\Omega_\delta)$ with the following properties:

$$\mathcal{E} \subset \Omega_\delta, \qquad \Omega_\delta \setminus \mathcal{E} \Subset M^n, \tag{4.28}$$

and $\hat{\Phi}$ satisfies (3.36) and (3.37) on Ω_δ. By applying Proposition 4.4 on Ω_δ, we obtain a positive function $v \in C^\infty(\Omega_\delta)$ satisfying (4.6), together with the asymptotic properties (4.17) and (4.18). Define $\hat{\rho} = \rho v$.

Proposition 4.6 (Existence of μ-bubble)**.** *Under the setting and hypotheses of Theorem 4.2, if*

$$m(\mathcal{E}, g) + \left(\tfrac{n-2}{n-1}\right)\beta \leq 0, \tag{4.29}$$

then there exists a μ-bubble $\Sigma^{n-1} \subset \Omega_\delta$ associated with $(\hat{\rho}, \hat{\Phi})$, which is smooth outside a closed singular set $\mathcal{S}_\Sigma$ residing within a compact subset of M^n, such that if it is nonempty

$$\dim_{\mathcal{M}} \mathcal{S}_\Sigma \leq n - 8. \tag{4.30}$$

Moreover, (Σ^{n-1}, g_Σ) satisfies

$$H_\Sigma = \hat{\Phi} - \nu(\log \hat{\rho}) \tag{4.31}$$

away from the singular set where ν is the unit outer normal, and it has a single harmonically asymptotically flat end $\mathcal{E}_\Sigma$, with the property that if $n \geq 4$ then the mass vanishes, $m(\mathcal{E}_\Sigma, g_\Sigma) = 0$.

Proof. It is enough to verify the barrier conditions along the boundary of the truncated regions; the rest of the construction is identical to the proof of Proposition 3.4. First, observe that using the expansions of ρ and v shows that

$$\hat{\rho} - \left(1 + \hat{\beta} r^{2-n}\right) \in C^{2,\alpha}_{2-n-q_1'}(\mathcal{E}), \qquad \hat{\beta} = \beta + \mathcal{C}. \tag{4.32}$$

Therefore, the assumed nonpositivity of the generalized mass, and the fact that $\mathcal{C} < 0$ by (4.18), gives the strict inequality

$$\mathfrak{m} := m(\mathcal{E}, g) + \frac{n-2}{n-1}\hat{\beta} = m(\mathcal{E}, g) + \frac{n-2}{n-1}\beta + \frac{n-2}{n-1}\mathcal{C} < 0. \tag{4.33}$$

We now verify the boundary barriers. Since $\hat{\Phi} = 0$ in the prescribed asymptotically flat end, the $(\hat{\rho}, \hat{\Phi})$-barrier quantity on a hypersurface with unit normal ν is

$$H + \nu(\log \hat{\rho}) - \hat{\Phi} = H + \nu(\log \hat{\rho}). \tag{4.34}$$

In the harmonically asymptotically flat coordinates, the horizontal slices $\{x^n = t\}$ satisfy, with respect to the normal $\nu = \eta\, \partial_{x^n} + o(1)$, where $\eta = \pm 1$,

$$H + \nu(\log \hat{\rho}) = -(n-1)\mathfrak{m}\, \eta t\, r^{-n} + O(r^{-n-q_1'}). \tag{4.35}$$

On the top face $\mathcal{P}_r^\sigma = \{x^n = \sigma\}$, the outward normal of $\mathcal{Z}_{r,\sigma}$ is asymptotic to $+\partial_{x^n}$, so $\eta t = \sigma > 0$. On the bottom face $\mathcal{P}_r^{-\sigma} = \{x^n = -\sigma\}$, the outward normal is asymptotic to $-\partial_{x^n}$, and again $\eta t = \sigma > 0$. Since $\mathfrak{m} < 0$, it follows that, after choosing σ and then r sufficiently large,

$$H + \nu(\log \hat{\rho}) - \hat{\Phi} > 0 \quad \text{ on } \mathcal{P}_r^{\pm\sigma} \cup \mathcal{D}_{r,\sigma}, \tag{4.36}$$

where the slow $\frac{1}{r}$-decay of the mean curvature is responsible for this inequality on the side cylinder $\mathcal{D}_{r,\sigma}$. □

The relevant stability inequality may be established in the same manner as Proposition 3.6, and is recorded here. Note that Corollary 4.5 guarantees that the scalar curvature quantity positivity hypothesis (3.61), needed for the proof of Proposition 3.6, is satisfied in the current setting.

Proposition 4.7 (Stability inequality)**.** *Consider the hypotheses and setting of Proposition 4.6, and let $\Sigma^{n-1} \subset \Omega_\delta$ be the constructed μ-bubble associated with $(\hat{\rho}, \hat{\Phi})$. For any $c \in \mathbb{R}$ let $f \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma)$ be a test function such that f vanishes in a neighborhood of $\mathcal{S}_\Sigma$, and $f - c \in W^{1,2}_{-\frac{n-3}{2}}(\mathcal{E}_\Sigma)$ when $n \geq 4$ or $\operatorname{supp}(f - c)$ is compact in $\overline{\mathcal{E}}_\Sigma$ when $n = 3$. Then there exists a constant $\delta_{n,\varepsilon} < \delta_0$ depending only on n and ε, and a positive function $\hat{Q} \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma) \cap C^{0,\alpha}_{-n+1-q_1'}(\mathcal{E}_\Sigma)$, such that for any $\delta \in (0, \delta_{n,\varepsilon})$ it holds that*

$$\begin{aligned}&\int_{\Sigma^{n-1}} \left(\hat{\rho} |\nabla_\Sigma f|^2_{g_\Sigma} + \frac{1}{2} \hat{\rho} \left(R_{g_\Sigma} - 2\Delta_{g_\Sigma} \log \hat{\rho} - \left(\frac{n-1}{n} + \varepsilon \right) |\nabla_\Sigma \log \hat{\rho}|^2_{g_\Sigma} \right) f^2 \right) dV_{g_\Sigma} \\ \geq &\int_{\Sigma^{n-1}} \hat{\rho} \hat{Q} f^2 dV_{g_\Sigma}.\end{aligned} \tag{4.37}$$

Combined with the blow-up procedure for GMT singularities in Section 3.3, the preceding construction yields a complete harmonically asymptotically Euclidean manifold with arbitrary ends carrying a weak $(n-1)$-data set.

Corollary 4.8. *Let $(M^n, g, \mathcal{E}, \rho, Q, \varepsilon)$ be a weak n-data set with $n \geq 4$. If the generalized mass satisfies*

$$m(\mathcal{E}, g) + \left(\tfrac{n-2}{n-1} \right) \beta \leq 0, \tag{4.38}$$

then there exists a weak $(n-1)$-data set $(M^{n-1}, g', \mathcal{E}', \rho', Q', \varepsilon')$ with vanishing mass

$$m(\mathcal{E}', g') = 0. \tag{4.39}$$

Proof. Let $(\Sigma^{n-1}, g_\Sigma, \mathcal{E}_\Sigma, \hat{\rho}, \hat{Q}, \varepsilon)$ be as in Proposition 4.7. Note that here $\hat{\rho}$ is restricted to Σ^{n-1}, and satisfies $\hat{\rho} - 1 \in C^{2,\alpha}_{3-n-q_1'}(\mathcal{E}_\Sigma)$. According to Proposition 4.6 the mass is zero $m(\mathcal{E}_\Sigma, g_\Sigma) = 0$, and for $\delta \in (0, \delta_{n,\varepsilon})$ it admits the stability inequality (4.37). If $n < 8$ then the singular set is empty $\mathcal{S}_\Sigma = \emptyset$, and thus this forms the desired weak $(n-1)$-data set.

Consider now the case when $n \geq 8$. The GMT singular set may be blown-up using the construction of Section 3.3. In particular let $w \in C^\infty(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma)$, with $w = 1$ on $\mathcal{E}_\Sigma$, be the function constructed in Lemma 3.9, and set

$$g' = w^{\frac{n+1}{n-3}} g_\Sigma, \qquad \rho' = w^{-\frac{n+1}{2}} \hat{\rho}, \qquad Q' = \frac{1}{2} w^{-\frac{n+1}{n-3}} \hat{Q}. \tag{4.40}$$

By Theorem 3.7, the blown-up manifold $(\Sigma^{n-1}\setminus\mathcal{S}_\Sigma, g')$ is complete and harmonically asymptotically flat with vanishing mass $m(\mathcal{E}_\Sigma, g')=0$. Moreover, it satisfies the stability inequality (3.69) if $\varepsilon\in(0,\varepsilon'_n)$, where we may take $\varepsilon'_n=\frac{1}{n(n+1)}$. Note also that $Q'\in C^{0,\alpha}_{1-n-q'_1}(\mathcal{E}_\Sigma)$ and $\rho'-1\in C^{2,\alpha}_{3-n-q'_1}(\mathcal{E}_\Sigma)$. Therefore, setting $M^{n-1}=\Sigma^{n-1}\setminus\mathcal{S}_\Sigma$, $\mathcal{E}'=\mathcal{E}_\Sigma$, and $\varepsilon'=\varepsilon$ yields the desired weak $(n-1)$-data set. $\square$

4.3. **Proof of Theorem 4.2.** We will first establish Theorem 4.2 in the 3-dimensional case. For general n, we will argue by contradiction and use Corollary 4.8 inductively to reduce the problem to dimension three.

Lemma 4.9. *Theorem 4.2 holds for $n=3$.*

Proof. Suppose to the contrary that $m(\mathcal{E},g)+\left(\frac{n-2}{n-1}\right)\beta\leq 0$. As in the proof of Corollary 4.8, we obtain a weak 2-data set $(\Sigma^2, g_\Sigma, E_\Sigma, \hat\rho, \hat Q, \varepsilon)$. In particular, for any $f\in C^1_c(\Sigma^2)$ the following stability inequality is satisfied

$$\int_{\Sigma^{n-1}}\left(\hat\rho|\nabla_\Sigma f|^2_{g_\Sigma}+\frac{1}{2}\hat\rho\left(R_{g_\Sigma}-2\Delta_{g_\Sigma}\log\hat\rho-\left(\frac{2}{3}+\varepsilon\right)|\nabla_\Sigma\log\hat\rho|^2_{g_\Sigma}\right)f^2\right)dV_{g_\Sigma}$$
$$\geq\int_{\Sigma^{n-1}}\hat\rho\hat Q f^2 dV_{g_\Sigma}. \tag{4.41}$$

We now rearrange this inequality so that it is adapted to the logarithmic cut-off argument. Thus, write $f=\hat\rho^{-1/2}\psi$ where $\psi\in C^1_c(\Sigma^2)$, integrate by parts, and apply Young's inequality to the mixed term to obtain

$$\int_{\Sigma^2}\left(4|\nabla_\Sigma\psi|^2_{g_\Sigma}+K_{g_\Sigma}\psi^2\right)dV_{g_\Sigma}\geq\int_{\Sigma^2}\hat Q\psi^2 dV_{g_\Sigma}, \tag{4.42}$$

where K_{g_Σ} denotes Gaussian curvature.

Following standard arguments, let x denote Cartesian coordinates on the asymptotically flat end $\mathcal{E}_\Sigma$, and for $r_0>1$ define the logarithmic cut-off function

$$\psi_r(x)=\begin{cases}1 & \text{if } |x|\leq r\\ 2-\frac{\log|x|}{\log r} & \text{if } r<|x|\leq r^2\\ 0 & \text{if } |x|\geq r^2\end{cases}. \tag{4.43}$$

Smoothing this function at the radii r and r^2 and using the same notation, we note that since the end has quadratic area growth

$$\int_{\Sigma^2}|\nabla_\Sigma\psi_r|^2_{g_\Sigma}dV_{g_\Sigma}\to 0\qquad\text{as } r\to\infty. \tag{4.44}$$

Combining this with the positivity of $\hat Q$, and the decay of the metric, it follows that

$$\int_{\Sigma^2}K_{g_\Sigma}dV_{g_\Sigma}>0. \tag{4.45}$$

On the other hand, let $D_r \subset \Sigma^2$ be the compact connected domain bounded by the coordinate circle $\{|x| = r\}$. Since the end $\mathcal{E}_\Sigma$ is asymptotically flat, the geodesic curvature of the boundary circle satisfies

$$\kappa_{g_\Sigma} = r^{-1} + o(r^{-1}), \qquad \int_{\partial D_R} \kappa_{g_\Sigma} ds = 2\pi + o(1). \tag{4.46}$$

Therefore, since the Euler characteristic admits the bound $\chi(D_r) \leq 1$, Gauss-Bonnet implies

$$\lim_{r\to\infty} \int_{D_r} K_{g_\Sigma} dV_{g_\Sigma} = \lim_{r\to\infty} \left(2\pi\chi(D_r) - \int_{\partial D_r} \kappa_{g_\Sigma} ds\right) \leq 0, \tag{4.47}$$

yielding a contradiction. □

Proof of Theorem 4.2 for $n > 3$. Suppose the contrary that the generalized mass satisfies

$$m(\mathcal{E}, g) + \left(\tfrac{n-2}{n-1}\right)\beta \leq 0. \tag{4.48}$$

Corollary 4.8 provides a weak $(n-1)$-data set $(M^{n-1}, g', \mathcal{E}', \rho', Q', \varepsilon')$ with vanishing mass, and strong decay for the weight function

$$m(\mathcal{E}', g') = 0, \qquad \rho' - 1 \in C^{2,\alpha}_{3-n-q_1'}(\mathcal{E}'). \tag{4.49}$$

In particular, the generalized mass of this new data set is zero. By repeating this dimensional reduction finitely many times, we arrive at a weak 3-data set with zero generalized mass, which contradicts Lemma 4.9. □

4.4. **Proof of the inequality in Theorem 1.1.** Proceeding by contradiction, assume that the mass of some asymptotically flat end is negative, $m(\mathcal{E}, g) < 0$. By Theorem 2.7, we may further assume that (M^n, g) has strictly positive scalar curvature $R_g > 0$ on $M^n \setminus \mathcal{S}$, and that all ends including $\mathcal{E}$ are harmonically asymptotically flat. By blowing-up the singular set as in Proposition 3.2, we obtain a complete manifold $(M^n \setminus \mathcal{S}, g')$ such that g' agrees with g on $\mathcal{E}$. Moreover, for each $\varepsilon \in (0, \varepsilon_n)$, there exists a positive function $\rho \in C^\infty(M^n \setminus \mathcal{S})$ such that $\rho = 1$ on $\mathcal{E}$ and

$$R_{g'} - 2\Delta_{g'} \log\rho - \left(\frac{n-1}{n} + \varepsilon\right) |\nabla \log\rho|^2_{g'} > 0 \quad \text{on } M^n \setminus \mathcal{S}. \tag{4.50}$$

Now apply Propositions 3.4 and 3.6 to obtain a μ-bubble $\Sigma^{n-1} \subset M^n \setminus \mathcal{S}$ associated with (ρ, Φ), that satisfies the stability inequality (3.62); here the barrier potential Φ is constructed in Lemma 3.3. Furthermore, (Σ^{n-1}, g_Σ) has a single asymptotically flat end $\mathcal{E}_\Sigma$ of zero mass, $m(\mathcal{E}_\Sigma, g_\Sigma) = 0$. If $n = 3$, we immediately obtain a contradiction between the stability inequality and the Gauss-Bonnet theorem, as in the proof of Lemma 4.9. If $n > 3$, then by blowing-up any potential GMT singularities of the μ-bubble, Theorem 3.7 gives rise to an $(n-1)$-data set $(\Sigma^{n-1} \setminus \mathcal{S}_\Sigma, g'_\Sigma, \mathcal{E}_\Sigma, \rho', Q', \varepsilon)$ with

$$m(\mathcal{E}_\Sigma, g'_\Sigma) = 0, \qquad \rho' = 1 \text{ on } \mathcal{E}_\Sigma. \tag{4.51}$$

In particular, the generalized mass of this data set vanishes, which yields a contradiction with Theorem 4.2. □

5. Rigidity Statement of the Singular PMT

The purpose of this section is to prove the rigidity statement in the equality case of Theorem 1.1. In the smooth setting, the vanishing of the ADM mass is usually treated by combining Ricci-flatness with harmonic coordinates and then applying the classical splitting or Bartnik-type rigidity argument. In the present singular setting, however, the lack of uniform Hessian control for harmonic functions near the singular set prevents a direct application of these methods. We overcome this difficulty by first proving Ricci-flatness on the regular set $M^n \setminus \mathcal{S}$, then constructing global harmonic coordinates with Hölder control near $\mathcal{S}$, and finally deriving a weighted Hessian estimate using cut-offs adapted to the Minkowski dimension bound on $\mathcal{S}$.

Throughout this section, it will be assumed for convenience that M^n has a single end. The case of additional ends may be treated with minor modifications.

5.1. **Ricci-flatness and harmonic coordinates.**

Lemma 5.1. *Let $(M^n, g, \mathcal{E})$ be a complete asymptotically flat L^∞-manifold. Suppose that the singular set $\mathcal{S}$ is compact and satisfies the Minkowski dimension upper bound, $\dim_{\mathcal{M}}\mathcal{S} < n-3+\frac{2}{n}$. If the scalar curvature of g is nonnegative on $M^n \setminus \mathcal{S}$, and the mass vanishes $m(\mathcal{E}, g) = 0$, then the Ricci curvature vanishes on $M^n \setminus \mathcal{S}$.*

Proof. We first establish a weak version of Lemma 5.1 for scalar curvature. Assume by way of contradiction that there is a point $p \in M^n \setminus \mathcal{S}$ with $R_g(p) > 0$. Then there is a small geodesic ball $B_{2r}(p) \Subset M^n \setminus \mathcal{S}$ on which the scalar curvature is strictly positive. Choose a cut-off function $\varphi \in C^\infty_c(B_{2r}(p))$ with $\varphi = 1$ on $B_r(p)$, and $0 \leq \varphi \leq 1$. Consider the following equation with prescribed asymptotics on the end $\mathcal{E}$:

$$\Delta_g u - \frac{n-2}{4(n-1)}\varphi R_g u = 0 \text{ on } M^n, \qquad u \to 1 \text{ as } r \to \infty. \tag{5.1}$$

Theorem 2.10 provides a positive solution $u \in C^\infty(M^n \setminus \mathcal{S}) \cap W^{1,2}(M^n, g) \cap L^\infty(M^n)$ such that

$$u - \left(1 + \frac{\mathcal{C}}{r^{n-2}}\right) \in C^{2,\alpha}_{2-n-q'}(\mathcal{E}) \qquad \mathcal{C} = -\frac{1}{4(n-1)\omega_{n-1}}\int_{M^n} \varphi R_g u \, dV_g < 0. \tag{5.2}$$

Moreover, u is uniformly bounded below by a positive constant. Then the conformal change $\hat{g} = u^{\frac{4}{n-2}} g$, yields a complete asymptotically flat L^∞-manifold $(M^n, \hat{g}, \mathcal{E})$ satisfying

$$R_{\hat{g}} = u^{-\frac{4}{n-2}}(1-\varphi)R_g \geq 0 \text{ on } M^n \setminus \mathcal{S}, \qquad m(\mathcal{E}, \hat{g}) = m(\mathcal{E}, g) + 2\mathcal{C} < 0, \tag{5.3}$$

giving a contradiction to the inequality statement of Theorem 1.1. Note that here we use the codimension thrshold of $3 - \frac{2}{n}$, in order to apply the positive mass inequality. We conclude that the scalar curvature vanishes, $R_g = 0$, away from the singular set.

We next show that $\mathrm{Ric}_g = 0$ on $M^n \setminus \mathcal{S}$, following the classic strategy of Schoen–Yau [43]. Suppose, to the contrary, that $\mathrm{Ric}_g(p) \neq 0$ for some $p \in M^n \setminus \mathcal{S}$. Choose a geodesic ball $B_{2r}(p) \Subset M^n \setminus \mathcal{S}$, and a cut-off function $\varphi \in C_c^\infty(B_{2r}(p))$ as above. For $t > 0$ sufficiently small, define

$$g_t := g - t\varphi \,\mathrm{Ric}_g \,. \tag{5.4}$$

Then $g_t = g$ outside $B_{2r}(p)$. In particular, g_t has the same asymptotics and the same ADM mass as g:

$$m(\mathcal{E}, g_t) = m(\mathcal{E}, g) = 0. \tag{5.5}$$

Recall the linearization of the scalar curvature

$$\left.\frac{d}{dt}\right|_{t=0} R_{g_t} = -DR_g(\varphi \,\mathrm{Ric}_g), \tag{5.6}$$

where

$$DR_g(h) = -\Delta_g(\mathrm{tr}_g\, h) + \mathrm{div}_g \,\mathrm{div}_g\, h - \langle \mathrm{Ric}_g, h\rangle_g. \tag{5.7}$$

Using that $R_g = 0$ and integrating by parts produces

$$\int_{M^n} R_{g_t} \, dV_{g_t} = t \int_{M^n} \varphi |\,\mathrm{Ric}_g\,|^2 \, dV_g + O(t^2) > 0 \tag{5.8}$$

for all sufficiently small $t > 0$.

Now solve

$$\Delta_{g_t} u_t - \frac{n-2}{4(n-1)} R_{g_t} u_t = 0 \text{ on } M^n, \qquad u_t \to 1 \text{ as } r \to \infty. \tag{5.9}$$

Note that for $t > 0$ sufficiently small, the negative part of R_{g_t} is small, and thus Theorem 2.10 applies to give a positive solution $u_t \in C^\infty(M^n \setminus \mathcal{S}) \cap W^{1,2}(M^n, g_t) \cap L^\infty(M^n)$ such that

$$u_t - \left(1 + \frac{\mathcal{C}_t}{r^{n-2}}\right) \in C^{2,\alpha}_{2-n-q'}(\mathcal{E}) \qquad \mathcal{C} = -\frac{1}{4(n-1)\omega_{n-1}} \int_{M^n} R_{g_t} u_t \, dV_{g_t}. \tag{5.10}$$

Since $u_t \to 1$ as $t \to 0$, the positivity of the total scalar curvature (5.8) implies that $\mathcal{C}_t < 0$ for all sufficiently small $t > 0$.

Next, define $\hat{g}_t := u_t^{\frac{4}{n-2}} g_t$ and observe that these metrics are scalar flat $R_{\hat{g}_t} = 0$, on $M^n \setminus \mathcal{S}$. Furthermore, $(M^n, \hat{g}_t, \mathcal{E})$ is again a complete asymptotically flat L^∞-manifold with mass

$$m(\mathcal{E}, \hat{g}_t) = m(\mathcal{E}, g_t) + 2\mathcal{C}_t = 2\mathcal{C}_t < 0. \tag{5.11}$$

This contradicts the inequality statement of Theorem 1.1. Hence, $\mathrm{Ric}_g = 0$ away from the singular set. □

On the asymptotically flat end $\mathcal{E}$, there are Cartesian coordinates $x = (x^1, \ldots, x^n)$ such that

$$g_{ij}(x) - \delta_{ij} = O_2(r^{-q}), \qquad \Delta_g x^i = O_1(r^{-1-q}), \qquad i,j = 1, \ldots, n, \tag{5.12}$$

where $q > \frac{n-2}{2}$. Thus each coordinate function x^i is asymptotically harmonic, and the error may be solved away to produce global harmonic functions y^i, referred to as *harmonic coordinates*, which play an important role in the rigidity argument.

Lemma 5.2. *Let $(M^n, g, \mathcal{E})$ be a complete asymptotically flat L^∞-manifold. Suppose that the singular set $\mathcal{S}$ is compact and satisfies the Minkowski dimension upper bound, $\dim_{\mathcal{M}} \mathcal{S} < n-2$. If the order of decay of this end satisfies $q \in (\frac{n-2}{2}, n-2)$, then there exist functions $y^i \in C^\infty(M^n \setminus \mathcal{S})$ such that*

$$\Delta_g y^i = 0 \text{ on } M^n, \qquad y^i - x^i \in C^{2,\alpha}_{1-q}(\mathcal{E}), \qquad i = 1, \ldots, n, \tag{5.13}$$

for $\alpha \in (0,1)$. Moreover, $y^i \in C^{0,\gamma}_{loc}(M^n)$ for some $\gamma \in (0,1)$, and there exists a constant $C > 0$ such that

$$|\nabla y^i(p)|_g \leq C \max\{d_{g_0}(p, \mathcal{S})^{\gamma-1}, 1\}, \qquad p \in M^n \setminus \mathcal{S}. \tag{5.14}$$

Proof. Choose a smooth cutoff function ζ which is equal to 1 on the asymptotic end outside a compact set and which vanishes on a neighborhood of the singular set. Set $\tilde{x}^i := \zeta x^i$. Then $\tilde{x}^i = x^i$ near infinity and, by the asymptotic flatness of the end,

$$\Delta_g \tilde{x}^i = O_1(r^{-q-1}) \quad \text{on } \mathcal{E}. \tag{5.15}$$

We solve

$$\Delta_g w^i = -\Delta_g \tilde{x}^i \quad \text{on } M^n, \tag{5.16}$$

with $w^i = o(r)$ on the asymptotic end. Since $\dim_{\mathcal{M}} \mathcal{S} < n-2$, the set $\mathcal{S}$ has zero $W^{1,2}$-capacity; hence no boundary condition is imposed along $\mathcal{S}$, and the weak equation across $\mathcal{S}$ is equivalent to the equation on $M^n \setminus \mathcal{S}$. The weighted elliptic theory on asymptotically flat manifolds [30, Theorem A.40], implies that $w^i \in C^{2,\alpha}_{1-q}(\mathcal{E})$. Define $y^i := \tilde{x}^i + w^i$, then this function is weakly harmonic on M^n, is smooth on $M^n \setminus \mathcal{C}$, and satisfies $y^i - x^i = w^i \in C^{2,\alpha}_{1-q}(\mathcal{E})$.

It remains to show the Hölder regularity of y^i and the gradient estimate near $\mathcal{S}$. Since (M^n, g) is an L^∞-manifold, the operator Δ_g may be written as a divergence-form elliptic operator with L^∞ coefficients that are uniformly elliptic. Standard De Giorgi–Nash–Moser theory [24, Theorem 8.9] applies, even across the singular set, to yield $\gamma \in (0,1)$ such that $y^i \in C^{0,\gamma}(M^n)$.

We now prove the desired gradient estimate. Because the metric g is smooth outside $\mathcal{S}$, standard gradient estimates for harmonic functions give a uniform bound for $|\nabla y^i|_g$ outside of $N_{r_0}(\mathcal{S})$, the closed r_0-neighborhood of $\mathcal{S}$ with respect to the background metric g_0, where $r_0 > 0$ is fixed. Take $p \in N_{r_0}(\mathcal{S}) \setminus \mathcal{S}$, and set $r_p := d_{g_0}(p, \mathcal{S}) > 0$. Then $B_{r_p/2}(p) \subset M^n \setminus \mathcal{S}$. Since the Ricci curvature vanishes in this ball, we can apply the Cheng–Yau gradient estimate [45, Corollary 3.2, Chapter I] to obtain

$$|\nabla y^i(p)|_g \leq \sup_{B_{r_p/4}} |\nabla(y^i - \inf_{B_{r_p/2}(p)} y^i)|_g \leq c r_p^{-1} \sup_{B_{r_p/2}(p)} |y^i - \inf_{B_{r_p/2}(p)} y^i|. \tag{5.17}$$

Furthermore, by the Hölder estimate above

$$\sup_{p', p'' \in B_{r_p/2}(p)} |y^i(p') - y^i(p'')| \leq c_0 d_{g_0}(p', p'')^\gamma \leq c_1 r_p^\gamma. \tag{5.18}$$

Combining (5.17) and (5.18) yields the desired result. We note that the final constant obtained is uniform, since $N_{r_0}(\mathcal{S})$ is compact. □

Define functions $g_{ij} = g(\partial_{y^i}, \partial_{y^j})$ and $g^{ij} = g(dy^i, dy^j)$, then by Bochner's formula

$$\frac{1}{2}\Delta_g g^{ii} = \frac{1}{2}\Delta_g |\nabla y^i|_g^2 = |\nabla^2 y^i|_g^2 + \mathrm{Ric}_g(\nabla y^i, \nabla y^i) = |\nabla^2 y^i|_g^2 \in C^{0,\alpha}_{-2-2q}(\mathcal{E}), \quad (5.19)$$

where $q \in (\frac{n-2}{2}, n-2)$. Therefore, the basic asymptotic analysis of [30, Corollary A.38] shows that

$$g^{ij} - \left(\delta_{ij} - \frac{c_{ij}}{r^{n-2}}\right) \in C^{2,\alpha}_{2-n-q_0}(\mathcal{E}) \quad (5.20)$$

for some set of constants $c_{ij} = c_{ji}$, where $q_0 > 0$. By performing a rotation of y-coordinates if necessary, it may be assumed without loss of generality that $c_{ij} = c_i\delta_{ij}$ for $i, j = 1, \ldots, n$. The metric satisfies the following asymptotics

$$g_{ij} = \left(1 + \frac{c_i}{r^{n-2}}\right)\delta_{ij} + O_2(r^{2-n-q_0}). \quad (5.21)$$

Corollary 5.3. *Under the hypotheses and setting of Lemma 5.2, it holds that*

$$m(\mathcal{E}, g) = \frac{(n-2)}{2n}\sum_{i=1}^n c_i. \quad (5.22)$$

Proof. In the harmonic coordinates $(y^1, \ldots, y^n)$, we have the following expansions

$$\sum_{i=1}^n \partial_i g_{ij} = (2-n)c_j r^{1-n}\nu^j + O(r^{1-n-q_0}), \quad (5.23)$$

and

$$\sum_{i=1}^n \partial_j g_{ii} = (2-n)\left(\sum_{i=1}^n c_i\right) r^{1-n}\nu^j + O(r^{1-n-q_0}), \quad (5.24)$$

where $\nu^j = \frac{y^j}{|y|}$. Therefore, the ADM mass flux density satisfies

$$\sum_{i,j=1}^n \big(\partial_i g_{ij} - \partial_j g_{ii}\big)\nu^j = (n-2)\left(\sum_{i=1}^n c_i - \sum_{i=1}^n c_i(\nu^i)^2\right) r^{1-n} + O(r^{1-n-q_0}). \quad (5.25)$$

Integrating over the coordinate sphere S_r and using

$$\int_{S_r} (\nu^i)^2 \, dA_\delta = \frac{1}{n}\omega_{n-1} r^{n-1}, \quad (5.26)$$

we obtain

$$\int_{S_r} \sum_{i,j=1}^n \big(\partial_i g_{ij} - \partial_j g_{ii}\big)\nu^j \, dA_\delta = \frac{(n-2)(n-1)}{n}\omega_{n-1}\sum_{i=1}^n c_i + o(1), \quad (5.27)$$

from which the desired result follows. □

5.2. **Proof of rigidity in Theorem 1.1.** Let $(M^n, g, \mathcal{E})$ be a complete asymptotically flat L^∞-manifold with a single end $\mathcal{E}$. Assume that the singular set $\mathcal{S}$ is compact and satisfies the Minkowski dimension upper bound, $\dim_{\mathcal{M}} \mathcal{S} \leq n-3+(n-1)^{-1}$, and that

$$R_g \geq 0 \text{ on } M^n \setminus \mathcal{S}, \qquad m(\mathcal{E}, g) = 0. \tag{5.28}$$

Note that since $(n-1)^{-1} < 2n^{-1}$, the hypotheses for the inequality portion of Theorem 1.1 are satisfied. According to Lemma 5.1, we have $\mathrm{Ric}_g = 0$ on $M^n \setminus \mathcal{S}$. From Lemma 5.2, there are global harmonic functions $\{y^i\}_{i=1}^n$ on M^n, and on $\mathcal{E}$ these functions form a coordinate system such that

$$g_{ij} - \left(1 + \frac{c_i}{r^{n-2}}\right)\delta_{ij} \in C^{2,\alpha}_{2-n-q_0}(\mathcal{E}). \tag{5.29}$$

The presence of the singular set $\mathcal{S}$ prevents us from obtaining the standard Hessian estimates for the harmonic coordinates y^i near $\mathcal{S}$. Consequently, the classical rigidity argument does not apply directly in the present setting. We overcome this difficulty by deriving new estimates which control the relevant gradient and Hessian terms. To this end, we introduce the following quantities

$$u^i = (|\nabla y^i|_g^2 + 1)^b, \qquad b = \frac{1}{2} - \frac{1}{2(n-1)} + \frac{n\epsilon}{2(n-1)}, \tag{5.30}$$

where $\epsilon > 0$ is an arbitrarily small parameter. A direct computation yields that on $M^n \setminus \mathcal{S}$,

$$\begin{aligned}
\Delta_g u^i =& b(|\nabla y^i|_g^2 + 1)^{b-1}\Delta_g|\nabla y^i|^2 + 4b(b-1)(|\nabla y_i|_g^2+1)^{b-2}|\nabla y^i|_g^2|\nabla|\nabla y^i|_g|_g^2 \\
\geq& 2b(|\nabla y^i|_g^2+1)^{b-1}\left(|\nabla^2 y^i|_g^2 + 2(b-1)|\nabla|\nabla y^i|_g|_g^2\right) \\
\geq& 2b(|\nabla y^i|_g^2+1)^{b-1}\left(1 + 2(b-1)\frac{n-1}{n}\right)|\nabla^2 y^i|_g^2 \\
=& 2b\epsilon(|\nabla y^i|_g^2+1)^{b-1}|\nabla^2 y^i|_g^2
\end{aligned} \tag{5.31}$$

where we have used the Bochner's formula and the refined Kato inequality

$$|\nabla^2 y^i|_g^2 \geq \frac{n}{n-1}|\nabla|\nabla y^i|_g|_g^2. \tag{5.32}$$

Choose the same cut-off function $\chi_r \in C_c^\infty(M^n)$ used in the proof of Proposition 2.8, which satisfies $0 \leq \chi_r \leq 1$ and additionally

$$\chi_r = 1 \text{ on } N_r(\mathcal{S}), \qquad \operatorname{supp}\chi_r \subset N_{2r}(\mathcal{S}) \qquad \int_{M^n} |\nabla\chi_r|_g^2 \leq C_\delta r^{1-\delta}, \tag{5.33}$$

where $\delta \in (\frac{1}{n-1}, 1)$. Set $r \leq r_0/4$ and $\eta_r = 1 - \chi_r$, where r_0 is as in the proof of Lemma 5.2. According to Lemma 5.2 we have

$$|\nabla y^i| \leq Cr^{\gamma-1} \quad \text{on } N_{2r}(\mathcal{S}) \setminus N_r(\mathcal{S}). \tag{5.34}$$

Multiplying both sides of (5.31) by η_r^2, integrating by parts, and applying Young's inequality produces

$$\begin{aligned}
&\int_{M^n} 2\epsilon b\eta_r^2(|\nabla y_i|_g^2+1)^{b-1}|\nabla^2 y^i|_g^2 dV_g \\
&= \lim_{R\to\infty}\int_{S_R}\nu(u^i)dA_g - 2\int_{N_{2r}(\mathcal{S})\setminus N_r(\mathcal{S})}\eta_r g(\nabla\eta_r,\nabla u^i)dV_g \qquad (5.35)\\
&\leq (n-2)c_i\omega_{n-1} + 4b\int_{N_{2r}(\mathcal{S})\setminus N_r(\mathcal{S})}\eta_r|\nabla\eta_r|_g(|\nabla y_i|_g^2+1)^{b-1}|\nabla y^i|_g|\nabla|\nabla y^i|_g|_g dV_g\\
&\leq (n-2)c_i\omega_{n-1} + \epsilon b\int_{M^n}\eta_r^2(|\nabla y_i|_g^2+1)^{b-1}|\nabla^2 y^i|_g^2 dV_g\\
&\quad + \frac{4b}{\epsilon}\int_{N_{2r}(\mathcal{S})\setminus N_r(\mathcal{S})}|\nabla\eta_r|_g^2(|\nabla y^i|_g^2+1)^b dV_g,
\end{aligned}$$

where ν is the unit outer normal to the coordinate sphere S_R, and we have used (5.20) in the third line. Next employ (5.33) and (5.34) to obtain

$$\begin{aligned}
\epsilon b\int_{M^n}\eta_r^2(|\nabla y_i|_g^2+1)^{b-1}|\nabla^2 y^i|_g^2 dV_g &\leq (n-2)c_i\omega_{n-1} + C_\epsilon r^{2b(\gamma-1)}\int_{M^n}|\nabla\chi_r|_g^2 dV_g\\
&\leq (n-2)c_i\omega_{n-1} + C_\epsilon C_\delta r^{2b(\gamma-1)+1-\delta}. \qquad (5.36)
\end{aligned}$$

Now choose ϵ sufficiently small, and δ sufficiently close to $\frac{1}{n-1}$ to achieve

$$2b(\gamma-1)+1-\delta>0. \qquad (5.37)$$

By sending $r\to 0$ it follows that

$$\epsilon b\sum_{i=1}^n\int_{M^n}(|\nabla y_i|_g^2+1)^{b-1}|\nabla^2 y^i|_g^2 dV_g \leq (n-2)\omega_{n-1}\sum_{i=1}^n c_i = 0, \qquad (5.38)$$

where we have used Corollary 5.3 and $m(\mathcal{E},g)=0$. Thus,

$$|\nabla^2 y^i|_g = 0 \quad \text{on } M^n\setminus\mathcal{S}, \quad i=1,\ldots,n. \qquad (5.39)$$

We now complete the rigidity argument. Since

$$\dim_{\mathcal{M}}\mathcal{S}\leq n-3+\frac{1}{n-1}<n-1, \qquad (5.40)$$

we have $\dim_{\mathcal{H}}\mathcal{S}<n-1$. By Szpilrajn's theorem [27, Chapter VII, Section 4], the topological dimension (an integer) of a metric space is bounded above by its Hausdorff dimension, so that $\dim_{\text{top}}\mathcal{S}\leq n-2$. According to the classical dimension-theoretic separation theorem, a closed subset of an n-manifold of topological dimension at most $n-2$ does not separate the manifold; see [20, Theorem 1.8.13]. Hence $M^n\setminus\mathcal{S}$ is connected. Moreover, in light of (5.39), the vector fields ∇y^i are parallel on $M^n\setminus\mathcal{S}$. It follows that the functions $g^{ij}:=g(\nabla y^i,\nabla y^j)$ are constant on $M^n\setminus\mathcal{S}$; note that connectedness is used here. Since $g^{ij}\to\delta^{ij}$ on the asymptotically flat end, we obtain

$$g(\nabla y^i,\nabla y^j)=\delta^{ij} \quad \text{on } M^n\setminus\mathcal{S}. \qquad (5.41)$$

Hence, the map

$$\Psi = (y^1, \dots, y^n) : M^n \to \mathbb{R}^n, \tag{5.42}$$

yields

$$g = \sum_{i=1}^{n} dy^i \otimes dy^i = \Psi^*\delta \quad \text{on } M^n \setminus \mathcal{S}. \tag{5.43}$$

In particular, $\Psi|_{M^n\setminus\mathcal{S}}$ is a smooth local isometry onto its image.

We now show that Ψ is surjective. Since $\Psi(x) = x + o(r)$ on the unique asymptotically flat end, Ψ is proper. Moreover, on each sufficiently large coordinate sphere $S_R \subset \mathcal{E}$, the map

$$x \mapsto \frac{\Psi(x)}{|\Psi(x)|} \tag{5.44}$$

is homotopic to the standard radial map $x \mapsto x/|x|$. Hence the induced boundary map has mod 2 degree one. By the relative degree formula, the proper continuous map $\Psi : M^n \to \mathbb{R}^n$ has mod 2 topological degree one. This does not require orientability of M or C^1-regularity of Ψ along $\mathcal{S}$. Thus Ψ is onto.

We next prove that Ψ is injective. Set

$$A := \mathbb{R}^n \setminus \Psi(\mathcal{S}), \qquad \mathcal{U} := \Psi^{-1}(A), \tag{5.45}$$

and note that $\mathcal{U} \subset M^n \setminus \mathcal{S}$. Thus, the restricted map $\Psi : \mathcal{U} \to A$ is a smooth local isometry. We first observe that A is connected. Indeed, after the vanishing of the Hessians, the functions y^i have uniformly bounded gradients with respect to a fixed smooth background metric. Hence Ψ is Lipschitz on M^n. Therefore

$$\dim_{\mathcal{H}} \Psi(\mathcal{S}) \le \dim_{\mathcal{H}} \mathcal{S} < n - 1. \tag{5.46}$$

By the dimension-theoretic separation theorem, $\Psi(\mathcal{S})$ does not disconnect $\mathbb{R}^n$. Thus $A = \mathbb{R}^n \setminus \Psi(\mathcal{S})$ is connected.

We claim that the restricted map is a global isometry. To see this, recall that $\Psi : \mathcal{U} \to A$ is a proper local homeomorphism. Moreover, since Ψ is surjective, this restriction map is onto. Therefore it is a covering map. On the asymptotically flat end, the functions y^i agree with the asymptotic coordinates up to lower-order terms. Hence, after passing sufficiently far out in the end, Ψ is one-to-one and its image contains the complement of a large compact set in $\mathbb{R}^n$. Thus the covering has one sheet near infinity. Since A is connected, the number of sheets of the covering is constant. Consequently, $\Psi : \mathcal{U} \to A$ is a one-sheeted covering, and yields a global isometry.

It remains to rule out the possibility that two distinct points of M^n map to a single point of $\Psi(\mathcal{S})$. Let

$$B := M^n \setminus \mathcal{U} = \Psi^{-1}(\Psi(\mathcal{S})). \tag{5.47}$$

Note that B is closed since $\Psi(\mathcal{S})$ is compact, as it is the image of the compact set $\mathcal{S}$ and Ψ is continuous. Moreover, $\dim_{\mathcal{H}} B < n-1$. Indeed, $B \cap \mathcal{S} \subset \mathcal{S}$, while on $M^n \setminus \mathcal{S}$ the map Ψ is a local diffeomorphism; hence, locally on the regular set, the preimage of $\Psi(\mathcal{S})$ has the same Hausdorff dimension as $\Psi(\mathcal{S})$, which is strictly less than $n-1$. Since g is uniformly equivalent to a smooth background metric, removing a closed set of Hausdorff dimension strictly less than $n-1$ does not change the metric completion. Thus the metric completion of $(\mathcal{U}, g)$ is (M^n, d_g). Similarly, the metric completion of (A, δ) is $(\mathbb{R}^n, d_\delta)$. The isometry $\Psi : \mathcal{U} \to A$ therefore extends uniquely to an isometry between the metric completions,

$$\Psi : (M^n, d_g) \to (\mathbb{R}^n, d_\delta). \tag{5.48}$$

This extension agrees with the original continuous map Ψ. This, of course, implies global injectivity, since if $p, q \in M^n$ satisfy $\Psi(p) = \Psi(q)$, then $d_g(p, q) = d_\delta(\Psi(p), \Psi(q)) = 0$.

We now have that $\Psi : M^n \to \mathbb{R}^n$ is a continuous bijection; in fact, as noted above Ψ is Lipschitz with respect to the background metric g_0, and the same is true of its inverse. Moreover, the extended map (5.48) is an isometry, so its inverse is also an isometry and is therefore continuous. Hence Ψ is a homeomorphism, and

$$\Psi|_{M^n \setminus \mathcal{S}} : M^n \setminus \mathcal{S} \longrightarrow \mathbb{R}^n \setminus \Psi(\mathcal{S}) \tag{5.49}$$

is a smooth isometry.

The case of multiple ends may be treated with minor modifications. We briefly indicate the steps here. If $\mathcal{E}_0$ is the zero-mass end, then the harmonic functions y^i may be chosen so that

$$y^i - x^i \to 0 \quad \text{on } \mathcal{E}_0, \qquad y^i \to 0 \quad \text{on every other end.} \tag{5.50}$$

Once it is established that $|\nabla^2 y^i|_g = 0$ on $M^n \setminus \mathcal{S}$, the quantities $g(\nabla y^i, \nabla y^j)$ are constant on the connected set $M^n \setminus \mathcal{S}$. Their limits are δ^{ij} on $\mathcal{E}_0$, but would be 0 on any other end, yielding a contradiction to the presence of additional ends. Thus the multiple-end case reduces to the single-end case considered above. $\square$

Example 5.1 (Failure of smooth removability under the L^∞ hypothesis)**.** The metric-space rigidity conclusion cannot in general be strengthened to smooth removability of the singular set under only an L^∞ assumption on the metric. To show this we provide an example.

Let $M^n = \mathbb{R}^n$ and $\mathcal{S} = \{0\}$. Choose a smooth diffeomorphism

$$\phi : S^{n-1} \to S^{n-1}, \tag{5.51}$$

which is isotopic to the identity but is not the restriction of a linear orthogonal transformation. Let ϕ_s, $s \in [0, 1]$, be a smooth isotopy from the identity to ϕ. Choose a smooth nonnegative cut-off function $\chi : [0, \infty) \to [0, 1]$ such that $\chi = 1$ for $r \leq 1$ and $\chi = 0$ for $r \geq 2$. Define a map

$$\Psi : \mathbb{R}^n \to \mathbb{R}^n \tag{5.52}$$

by $\Psi(0) = 0$ and, for $x = (r, \theta)$, $r > 0$, $\theta \in S^{n-1}$,

$$\Psi(r, \theta) = r\, \phi_{\chi(r)}(\theta). \tag{5.53}$$

Then Ψ is a bi-Lipschitz homeomorphism of $\mathbb{R}^n$, is a smooth diffeomorphism on $\mathbb{R}^n \setminus \{0\}$, and agrees with the identity map outside a compact set. However, Ψ

is not C^1 at the origin: if $D\Psi(0)$ existed, then

$$\lim_{r\to 0}\frac{\Psi(r,\theta)}{r}=\phi(\theta) \tag{5.54}$$

would have to be the restriction to S^{n-1} of a linear map, contrary to the choice of ϕ.

Now define

$$g:=\Psi^*\delta \quad \text{on } \mathbb{R}^n\setminus\{0\}. \tag{5.55}$$

Then g is smooth and flat on $M^n\setminus\mathcal{S}$, is uniformly equivalent to the Euclidean metric, and hence defines an L^∞-metric on M^n. Moreover, since $\Psi=\mathrm{Id}$ outside a compact set, the metric g is asymptotically flat with zero ADM mass, and

$$R_g=0 \quad \text{on } M^n\setminus\mathcal{S}. \tag{5.56}$$

The singular set satisfies $\dim_{\mathcal{M}}\mathcal{S}=0$. Furthermore,

$$\Psi:(\mathbb{R}^n\setminus\{0\},g)\longrightarrow(\mathbb{R}^n\setminus\{0\},\delta) \tag{5.57}$$

is a smooth isometry, and Ψ extends to a homeomorphism of the metric completions.

Nevertheless, g does not extend smoothly across the origin in the original smooth structure. Indeed, near the origin

$$g=dr^2+r^2\phi^*g_{S^{n-1}}. \tag{5.58}$$

If g extended smoothly, as a Riemannian metric at the origin in the given smooth structure, the limiting inner product at the origin would determine the standard round metric on infinitesimal spheres. This would force

$$\phi^*g_{S^{n-1}}=g_{S^{n-1}}, \tag{5.59}$$

so that ϕ is an isometry of the round sphere, contradicting the choice of ϕ. Thus the singularity is removable in the metric-space sense, but not necessarily in the smooth sense.

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