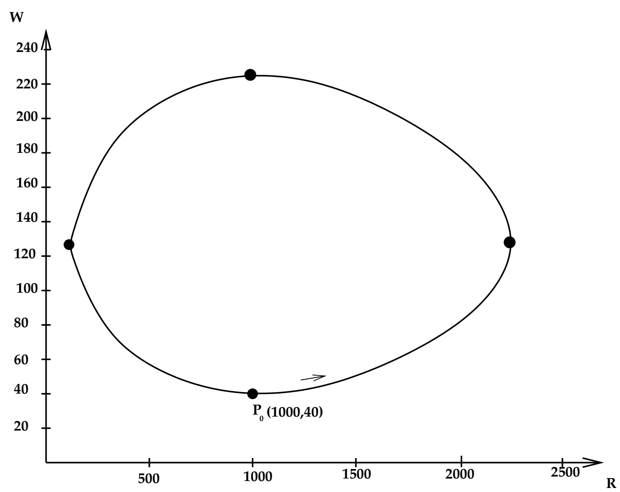
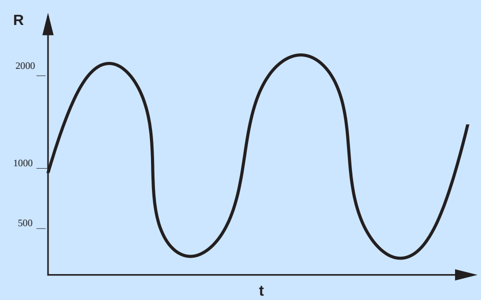
# Solutions to MAT127 Fall'05 Midterm 2

1. The graph below shows a trajectory in the phase plane for a certain predator-prey model. $R$ denotes the number of rabbits and $W$ denotes the number of wolves. Initially (at time $t = 0$), $R = 1000$ and $W = 40$.




   (a) (10 points) Sketch a rough graph of $R$ as a function of $t$ =time.

   **Solution:** Here is a *rough* sketch:




   (b) (5 points) When the number of rabbits reaches its global maximum, about how many wolves are there?

   **Solution:** From the phase portrait, we see the number of rabbits is largest (that is, the curve goes furthest to the right) at about (2450,130). Hence the number of wolves should be about 130.

   (c) (5 points) When the number of rabbits reaches its global maximum, is the wolf population increasing or decreasing?

   **Solution:** Since the curve in the phase plane is traversed counter-clockwise, the number of wolves in increasing then.

2. (10 points) Give an example of a sequence which is bounded and diverges. Explain why!

**Solution:** There are many answers, but one straightforward answer is the sequence

$$\{(-1)^n\}_{n=0}^{\infty} = 1, -1, 1, -1, 1, -1, \ldots$$

This sequence is bounded, since $-1 \leq a_n \leq 1$ for all $n$, but it diverges since $\lim_{n\to\infty} a_n$ does not exist.

3. Determine whether the following sequences are convergent or not. If convergent compute their limits. Show your work!

   (a) (10 points)

$$a_n = \frac{\sqrt{n}}{1+\sqrt{n+1}}$$

**Solution:**

$$\lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n+1}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}} + \sqrt{\frac{n+1}{n}}} = \frac{1}{0+1} = 1$$

There are also other methods you could use.

   (b) (10 points)

$$a_n = \frac{2^n}{3^{n+1}}$$

**Solution:**

$$\lim_{n\to\infty} \frac{2^n}{3^{n+1}} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 0$$

4. (a) (10 points) Express the number $0.\overline{32} = 0.323232323232\ldots$ as a ratio of two integers.

**Solution:**

$$0.\overline{32} = 32\sum_{n=1}^{\infty}\left(\frac{1}{100}\right)^n = 32\left(\frac{1}{1-1/100} - 1\right) = \frac{32}{99}$$

   (b) (10 points) Evaluate the sum $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \frac{5}{81} + \cdots$

**Solution:**

$$\frac{5}{\sum_{n=0}}^{\infty}\left(\frac{1}{3}\right)^n = \frac{5}{1-1/3} = \frac{15}{2}$$

(c) (10 points) Determine whether the following series converges or not (explain why!)

$$\sum_{n=1}^{\infty} \frac{5}{3^n - n}$$

**Solution:** This looks a lot like the geometric series $\sum \frac{1}{3^n}$. However, since $1/(3^n - n) > 1/3^n$ we can't directly compare the two, and we haven't yet learned the limit comparison test (use it if you know it). Instead, we have to be a little smarter. Notice that for $n \geq 1$,

$$2^n \leq 3^n - 1, \quad \text{and so} \quad \frac{1}{3^n - n} \leq \frac{1}{2^n}.$$

This means we have

$$\sum_{n=1}^{\infty} \frac{5}{3^n - n} < 5 \sum_{n=1}^{\infty} \frac{1}{2^n} = 5$$

and so the series converges.

5. (a) (10 points) Does the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n} = \frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln(4)}{4} + \cdots$ converge or diverge? Explain why.

**Solution:** This series diverges.

Since $\dfrac{\ln n}{n} > \dfrac{1}{n}$ for $n > 3$, we know that

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} > \sum_{n=3}^{\infty} \frac{1}{n}$$

But the latter sum is the harmonic series starting at n=3, which diverges. Putting the first term back into our series only makes it bigger, so it still diverges, since all terms but the first are larger than the terms of a divergent series with positive terms.

(b) (10 points) Does $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \frac{1}{4(\ln 4)^2} + \cdots$ converge or diverge? Explain why.

**Solution:** We'll use the integral test.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^{\infty} \frac{du}{u^2} = \lim_{N \to \infty} \left( \frac{-1}{N} - \frac{-1}{\ln 2} \right) = \frac{1}{\ln 2}$$

where we made the substitution $u = \ln x$ (so $du = \frac{dx}{x}$) to do the integral.

Since the integral is finite, the original series must converge.

The question doesn't ask us, but the sum of the series is no more than the first term plus the value of the integral, that is

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} < \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}$$