Midterm Exam: MAT 342

Instructions: Complete all problems below. You may not use calculators or other electronic devices, including cell phones. Show all of your work. **Be sure to write your name and student ID on each page that you hand in.**

1.(13pts) Let $z_1 = -1 + i$ and $z_2 = \sqrt{2}e^{\frac{\pi}{4}i}$.

a) Calculate $\overline{z}_1 z_2$, and write your answer in the form $a + bi$.

b) Calculate $z_1^{1/3}$ and sketch the solution set in the complex plane.

2.(13pts) Determine if the following limits exist or not. If they exist, find the limit:

$$\text{a)} \quad \lim_{z \to i} \frac{\overline{(z-i)^2}}{(z-i)^2}, \qquad \text{b)} \quad \lim_{z \to e^{\frac{\pi}{4}i}} \frac{z^2 - i}{z^4 + 1}.$$

You must justify all of your work. Simply stating an answer will receive no credit.

3.(15pts) a) Sketch the region given by $-\frac{\pi}{6} < \text{Arg} z \leq 0$, $1 < |z| < 2^{1/3}$.

b) Find the image of the above region under the mapping $w = z^3$.

4.(14pts) Let $f(z) = e^{\frac{z+1}{z-1}}$. Find the domain of this function, explain why it is holomorphic on its domain, and calculate $f'(z)$.

5.(15pts) If $g(z)$ is a holomorphic function on a domain D, and $\text{Re}(g(z))$ is constant on D, what can you say about $g(z)$? Explain your reasoning in detail.

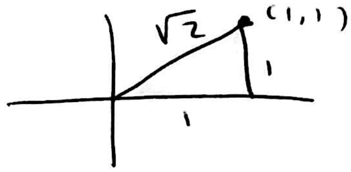
6.(15pts) Let $z = x+iy$. Find the points where the function $f(z) = (x^2+y^2)+(x-y)i$ is differentiable, and then calculate $f'(z)$ at these points. Is the function holomorphic at these points? Give full explanations.

7.(15pts) Choose the following branch of the logarithm:

$$\log z = \ln|z| + i\theta, \qquad -\frac{5\pi}{3} \leq \theta < \frac{\pi}{3}.$$

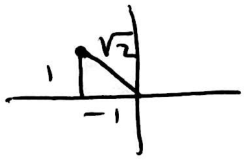
Calculate $\log(1+i)^2$ and $2\log(1+i)$ with respect to this branch. Are they equal to each other?

1. a) $\bar{z}_1 z_2 = (-1-i)\sqrt{2}\, e^{\frac{\pi}{4} i}$



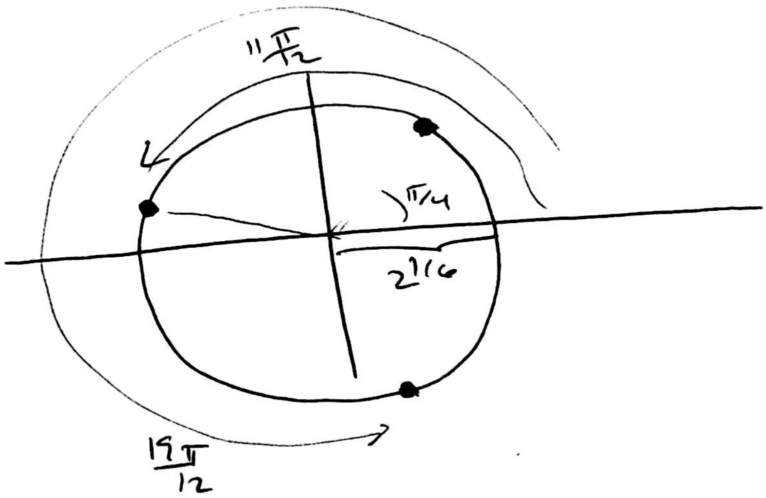
$= (-1-i)(1+i)$

$= -1 - i - i + 1 = -2i$



b) $z_1^{1/3} = \left(\sqrt{2}\, e^{\left(\frac{3\pi}{4} + 2n\pi\right) i}\right)^{1/3} = 2^{1/6}\, e^{\left(\frac{\pi}{4} + \frac{2n\pi}{3}\right) i}$

$= \left\{ 2^{1/6} e^{\frac{\pi}{4} i},\ 2^{1/6} e^{\frac{11}{12}\pi i},\ 2^{1/6} e^{\frac{19}{12}\pi i} \right\}$



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2 a) $\lim\limits_{z \to i} \frac{\left(\overline{z-i}\right)^2}{(z-i)^2} = \lim\limits_{z_1 \to 0} \frac{\bar{z}_1^{\,2}}{z_1^{\,2}} = \lim\limits_{z_2 \to 0} \frac{\bar{z}_2}{z_2}$

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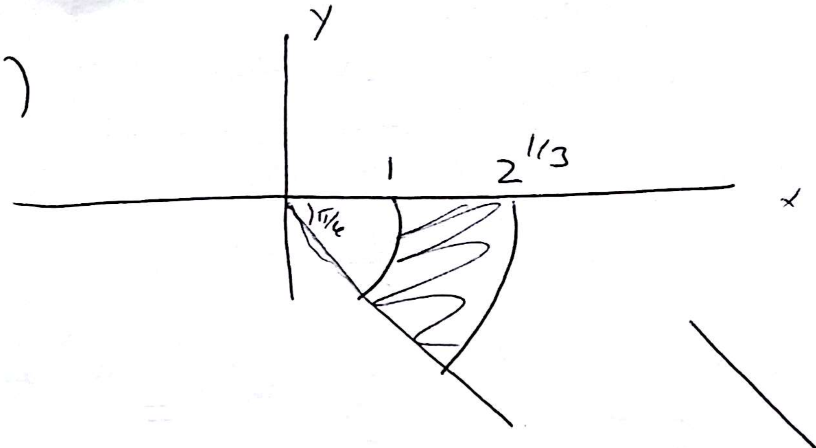
To prove this consider limits on vertical & horizontal lines. Show they do not agree.

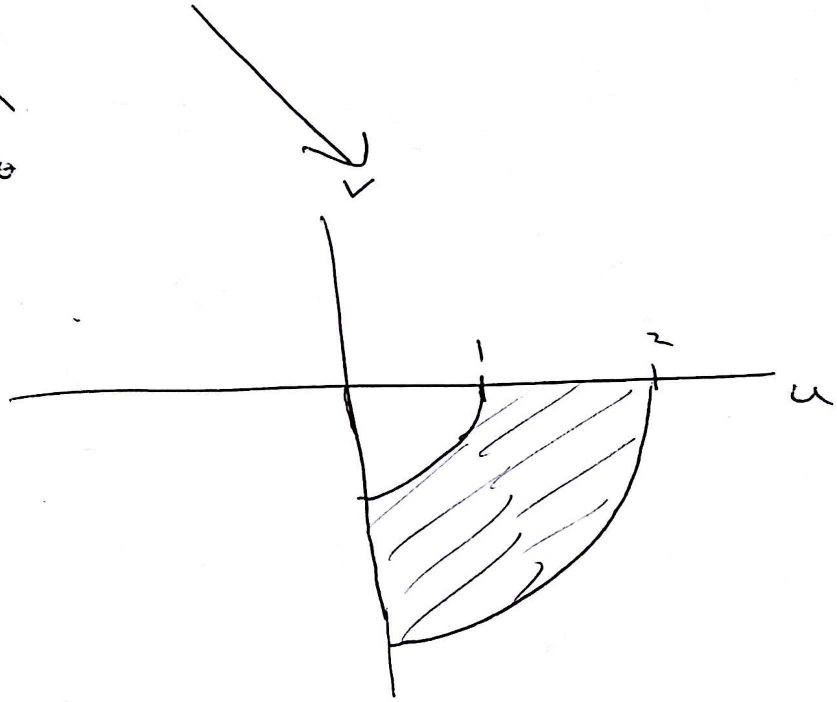
b) $\lim\limits_{z \to e^{\frac{\pi}{4} i}} \left[\frac{z^2 - i}{z^4 + 1} = \frac{z^2 - i}{(z^2+i)(z^2-i)} \right] = \lim\limits_{z \to e^{\frac{\pi}{4} i}} \frac{1}{z^2 + i}$

$= \frac{1}{e^{\pi/2\, i} + i} = \boxed{\frac{1}{2i}}$

3. a)



b) $w = z^3 = r^3 e^{3i\theta}$



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4. Domain $= \{ z \neq 1 \}$.

It is holomorphic since it is the composition of holomorphic functions here.

$$f' = e^{\frac{z+1}{z-1}} \left(\frac{1}{z-1} - \frac{(z+1)}{(z-1)^2} \right)$$

$$\underbrace{\phantom{\frac{1}{z-1} - \frac{(z+1)}{(z-1)^2}}}_{} \quad \frac{\cancel{z}-1 - (\cancel{z}+1)}{(z-1)^2} = \frac{-2}{(z-1)^2}$$

$$f' = -\frac{2}{(z-1)^2} e^{\frac{z+1}{z-1}}$$

5. $\boxed{f(z) = \text{const.}}$

Proof: $f(z) = u + iv$. $u = \text{const.}$

$$\Rightarrow \begin{matrix} 0 = u_x = v_y \\ 0 = u_y = -v_x \end{matrix} \Rightarrow v = \text{const.}$$

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6. $f = x^2 + y^2 + (x-y)i$

$u_x = 2x, \quad u_y = 2y, \quad v_x = 1, \quad v_y = -1$

$u_x = v_y \Rightarrow x = -\frac{1}{2}$

$u_y = -v_x \Rightarrow y = -\frac{1}{2}$

$f'(-\frac{1}{2} - \frac{1}{2}i) = u_x + iv_x = -1 + i$

So f is differentiable only at $(-\frac{1}{2}, -\frac{1}{2})$.

Not holomorphic there since need a neighborhood for this to be true.

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7. $\log(1+i)^2 = \log(\sqrt{2}e^{i\pi/4})^2 = \log 2e^{i\pi/2} = \boxed{\ln 2 + i(-\frac{3\pi}{2})}$

$2\log(1+i) = 2\log\sqrt{2}e^{i\pi/4} = 2(\ln\sqrt{2} + i\pi/4)$

$= \boxed{\ln 2 + i\pi/2}$ Not equal.