MAT 127: Calculus C, Spring 15

Solutions to Midterm II

Problem 1 (15pts)

(a; 7pts) *Show that the function* $y(x) = -\mathrm{e}^{-x}\sin x$ *is a solution to the initial-value problem*

$$y'' + 2y' + 2y = 0, \quad y = y(x), \qquad y(0) = 0, \qquad y'(0) = -1.$$

Show your work and/or explain your reasoning.

Compute $y'(x)$ and $y''(x)$ to check that the ODE is satisfied:

$$\begin{aligned}
y(x) = -\mathrm{e}^{-x}\sin x \quad &\Longrightarrow \quad y'(x) = -\big(-\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x\big) = \mathrm{e}^{-x}\sin x - \mathrm{e}^{-x}\cos x \\
&\Longrightarrow \quad y''(x) = -\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x - \big(-\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x\big) = 2\mathrm{e}^{-x}\cos x \\
&\Longrightarrow \quad y'' + 2y' + 2y = 2\mathrm{e}^{-x}\cos x + 2\big(\mathrm{e}^{-x}\sin x - \mathrm{e}^{-x}\cos x\big) + 2\big(-\mathrm{e}^{-x}\sin x\big) = 0. \checkmark
\end{aligned}$$

It remains to check that the initial condition are satisfied:

$$y(0) = -\mathrm{e}^{-0}\sin 0 = -1\cdot 0 = 0, \checkmark \qquad y'(0) = \mathrm{e}^{-0}\sin 0 - \mathrm{e}^{-0}\cos 0 = 1\cdot 0 - 1\cdot 1 = -1 \checkmark$$

Grading: $y'(x)$ 1pt; $y''(x)$ from $y'(x)$ 2pts; plug in into DE and simplify 1pt each; check initial conditions 1pt each; no carry-over penalties

(b; 8pts) *Find the general solution of the differential equation*

$$y'' + 2y' + 2y = 0, \quad y = y(x).$$

Show your work and/or explain your reasoning.

Since $y(x) = -\mathrm{e}^{-x}\sin x$ is a solution of this second-order linear homogeneous ODE with constant real coefficient

$$y(x) = \mathrm{e}^{-x}\sin x \qquad \text{and} \qquad y(x) = \mathrm{e}^{-x}\cos x$$

are also solutions (by the structure theorem for solutions of such equations). Furthermore, the general solution of the differential equation is

$$\boxed{y(x) = C_1\mathrm{e}^{-x}\cos x + C_2\mathrm{e}^{-x}\sin x}$$

Alternatively, the associated polynomial equation for this differential equation is

$$r^2 + 2r + 2 = 0 \qquad \Longrightarrow \qquad r_1, r_2 = -1 + \sqrt{1-2} = -1 + 1\mathrm{i}.$$

This is the complex case with $p=-1$ and $q=1$, which again gives the above general.

Grading: in the 2nd approach, 3pts for the roots, 2pts each for $y(x)$ containing $\mathrm{e}^{-x}\sin x$ and $\mathrm{e}^{-x}\cos x$ (only 1pt each for $\mathrm{e}^{(-1+\mathrm{i})x}$ and $\mathrm{e}^{(-1-\mathrm{i})x}$), with remainder for the correct formula; if the roots are obtained incorrectly, 1pt for setting up the polynomial and up to 4 additional points for converting the roots to the corresponding equation for $y(x)$; in the 1st approach, a little bit of explanation is required (anything in parentheses above is not required); if no explanation is given, 4pts for correct answer, 2pts if $y(x)$ contains only one constant *or* missing $\mathrm{e}^{-x}\cos x$ or $\mathrm{e}^{-x}\sin x$; 1pt for $y(x) = C\mathrm{e}^{-x}\cos x$ or $y(x) = C\mathrm{e}^{-x}\sin x$

Solution to this problem continues on the next page.

Alternatively, one could first use the second approach in (b) to find that the general solution to the differential equation is

$$y(x) = C_1 \mathrm{e}^{-x} \cos x + C_2 \mathrm{e}^{-x} \sin x$$

and then find C_1 and C_2 so that the two initial conditions in (a) hold. In order to do this, compute $y'(x)$, $y(0)$, and $y'(0)$ and set them equal to the given initial condition:

$$\begin{aligned}
y'(x) &= C_1\big(-\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x\big) + C_2\big(-\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x\big)\\
&= (C_2 - C_1)\,\mathrm{e}^{-x}\cos x - (C_1{+}C_2)\,\mathrm{e}^{-x}\sin x\,,\\
y(0) &= C_1\mathrm{e}^{-0}\cos 0 + C_2\mathrm{e}^{-0}\sin 0 = C_1\\
y'(0) &= (C_2 - C_1)\,\mathrm{e}^{-0}\cos 0 - (C_1{+}C_2)\,\mathrm{e}^{-0}\sin 0 = C_2 - C_1\,,\\
&\Longrightarrow \quad \begin{cases} y(0) = C_1 = 0 \\ y'(0) = C_2 - C_1 = -1 \end{cases} \qquad \Longrightarrow \qquad C_1 = 0,\ C_2 = -1.
\end{aligned}$$

So the solution to the initial-value problem in (a) is

$$y(x) = 0 \cdot \mathrm{e}^{-x}\cos x - 1 \cdot \mathrm{e}^{-x}\sin x = -\mathrm{e}^{-x}\sin x\,,$$

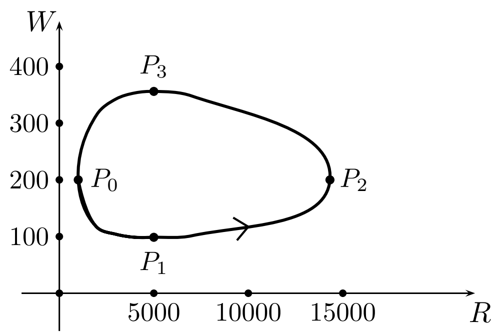
which is the function given in (a).

Grading: finding the general solution as in the second approach to part (b) on the previous page; after that, finding $y'(x)$ 2pts, setting up the system 2pts; finding C_1 and C_2 2pts, and concluding the argument 1pt.

Remark: using the last approach indicates solid understanding of how to solve initial-value problems, but also misunderstanding of the basic concept of how to check that a given function solves a given initial-value problem.

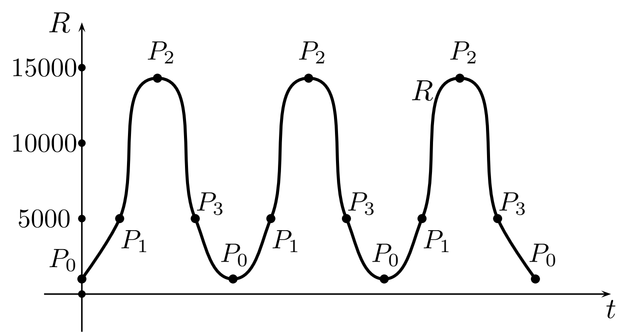
Problem 2 (20pts)

The graph on the left shows a trajectory in the phase plane for the predator-prey model described by the system of ODEs on the right. R denotes the number of rabbits and W denotes the number of wolves. Initially (at time $t = 0$), $R = 1000$ and $W = 200$.



$$\begin{cases} \frac{\mathrm{d}R}{\mathrm{d}t} = R - \frac{1}{200}RW \\ \frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{1}{10}W + \frac{1}{50{,}000}RW \end{cases}$$

(a; 10pts) *Sketch a rough graph of R as a function of t = time.*



First, determine the key points on the phase trajectory in the order they are traversed as t increases (counter-clockwise in the above diagram). These are

$$P_0 = (1000, 200), \quad P_1 \approx (5000, 100), \quad P_2 \approx (15000, 200), \quad P_3 \approx (5000, 350).$$

Mark the first coordinate of each of the key points of the trajectory on a diagram with horizontal t-axis and a vertical R-axis in the same order. Connect them by a smooth curve with the only minimum at P_0 and the only maximum at P_2. Since the phase trajectory is periodic, so is the graph of R; so repeat this curve several times.

(b; 2pts) *When the number of rabbits reaches its global maximum, about how many wolves are there? Answer only.*

This is the W-coordinate of the right-most point on the graph, i.e. of P_2: $\boxed{200}$

(c; 2pts) *When the number of rabbits reaches its global maximum, is the wolf population increasing or decreasing? Answer only.*

From P_2, the function W is $\boxed{\text{increasing}}$ which makes sense (lots of rabbits is beneficial to wolves).

(d; 6pts) *Find the equilibrium solutions of the system of ODEs.*

The equilibrium (constant) solutions are the pairs of numbers (R, W) such that

$$\begin{cases} \frac{\mathrm{d}R}{\mathrm{d}t} = 0 \\ \frac{\mathrm{d}W}{\mathrm{d}t} = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} R(1 - \frac{1}{200}W) = 0 \\ -\frac{1}{10}W(1 - \frac{1}{5{,}000}R) = 0 \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} R = 0 \text{ or } W = 200 \\ W = 0 \text{ or } R = 5,000 \end{cases}$$

Choosing one condition from the first line and one from the second, we obtain $2\times2=4$ systems of linear equations

$$\begin{cases} R = 0 \\ W = 0 \end{cases} \qquad \begin{cases} R = 0 \\ R = 5,000 \end{cases} \qquad \begin{cases} W = 200 \\ W = 0 \end{cases} \qquad \begin{cases} W = 200 \\ R = 5,000 \end{cases}$$

The second and third systems of equations have no solutions, while the first and the forth give us

$$\boxed{(R, W) = (0, 0), (5000, 200)}$$

Note that the coordinates of the second equilibrium correspond to the R-coordinate of the W-minimum and maximum on the phase trajectory (i.e P_1 and P_3) and the W-coordinate of the R-minimum and maximum on this trajectory (i.e P_0 and P_2).

Problem 3 (20pts)

Write your answer to each question in the corresponding box in the simplest possible form. Justify your answers in the spaces below.

(a; 7pts) *Find the limit of the sequence* $a_n = n^2\big(1-\cos(1/n)\big)$.

$$\boxed{\frac{1}{2}}$$

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{1-\cos(1/n)}{1/n^2} = \lim_{x\longrightarrow 0} \frac{1-\cos(x)}{x^2} = \lim_{x\longrightarrow 0} \frac{0+\sin(x)}{2x} = \lim_{x\longrightarrow 0} \frac{\cos(x)}{2} = \frac{1}{2}.$$

The third and fourth equalities use l'Hospital, which is applicable here because $1-\cos(x), x^2 \longrightarrow 0$ and $\sin(x), x \longrightarrow 0$ as $x \longrightarrow 0$.

(b; 7pts) *Find the limit of the sequence*

$$\boxed{2}$$

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ \ldots$$

Here is a quick approach that works in this case:

$$a_n = 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdot \ldots 2^{\frac{1}{2^n}} = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}} = 2^{1-\frac{1}{2^n}} = 2^1 \cdot 2^{-\frac{1}{2^n}} \longrightarrow 2 \cdot 2^{-\frac{1}{\infty}} = 2 \cdot 1 = 2$$

Here is another approach that works more generally. This sequence is recursively defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$. If it converges to some number a, then

$$a = \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} a_{n+1} = \lim_{n\longrightarrow\infty} \sqrt{2a_n} = \sqrt{2 \lim_{n\longrightarrow\infty} a_n} = \sqrt{2a};$$

so $a = \sqrt{2a}$ or $a^2 - 2a = 0$. This gives $a = 2, 0$; since $a_n \geq 1$ for all n (square root of two numbers greater than 1 is greater than 1), the limit $a = 2$ *provided* it exists at all.

(c; 6pts) *Write the number* $1.0\overline{50} = 1.0505050\ldots$ *as a simple fraction.*

$$\boxed{\frac{104}{99}}$$

$$1.0\overline{50} = 1 + \frac{5}{100} + \frac{5}{100^2} + \ldots = 1 + \frac{5/100}{1-\frac{1}{100}} = 1 + \frac{5/100}{99/100} = 1 + \frac{5}{99} = \frac{104}{99}$$

Problem 4 (10pts)

Determine whether each of the following sequences converges or diverges. In each case, circle your answer to the right of the question and justify it in the space provided below the question. If the sequence converges, find its limit.

(a; 5pts) $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$ **converge** **diverge**

Divide the top and bottom of the fraction by the highest power of n:

$$a_n = (-1)^n \frac{n^3/n^3}{n^3/n^3 + 2n^2/n^3 + 1/n^3} = (-1)^n \frac{1}{1+2/n+1/n^3}\,.$$

As $n \longrightarrow \infty$, the above fraction approaches

$$\frac{1}{1+2/n+1/n^3} \longrightarrow \frac{1}{1+2/\infty+1/\infty^3} = \frac{1}{1+0+0} = 1.$$

However, $(-1)^n$ is 1 when n is even and -1 when n is odd. So, the terms a_n with n even converge to 1, while the terms a_n with n odd converge to -1. Thus, the entire sequence a_n diverges (it keeps on jumping between near 1 and near -1; "2 limits" means "no limit").

Grading: wrong answer 0pts regardless of explanation; correct answer *circled* 1pt; reasonable explanation up to 4pts (anything in parenthesis not required)

(b; 5pts) $\left\{1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}, \dots\right\}$ **converge** **diverge**

This is the sequence of partial sums for the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$:

$$s_n = \sum_{k=1}^{n}\frac{1}{k}\,.$$

The harmonic series is a p-Series with $p=1$ and thus diverges by the p-Series Test. By definition, this means that the sequence of partial sums s_n diverges.

Grading: wrong answer 0pts regardless of explanation; correct answer *circled* 1pt; reasonable explanation up to 4pts

Problem 5 (15pts)

(a; 7pts) *Determine whether the following series* **converges** or **diverges**

$$\sum_{n=1}^{\infty} ne^{-n}$$

Circle your answer above and justify it below.

The quickest way here is to use the *Ratio Test* of Section 8.4 (because of e^{-n}):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)\mathrm{e}^{-(n+1)}}{n\mathrm{e}^{-n}} = \left(1+\frac{1}{n}\right)\cdot\frac{\mathrm{e}^{-n}\mathrm{e}^{-1}}{\mathrm{e}^{-n}} = \left(1+\frac{1}{n}\right)\mathrm{e}^{-1} \longrightarrow \left(1+\frac{1}{\infty}\right)\mathrm{e}^{-1} = \mathrm{e}^{-1}.$$

Since $\mathrm{e}^{-1} = 1/\mathrm{e} < 1$, the series converges.

We can also use the *Limit Comparison Test*. Note that $0 < \mathrm{e}^{-n/2}$, $\sum_{n=1}^{\infty} \mathrm{e}^{-n/2}$ converges being a geometric series with $r = 1/\sqrt{\mathrm{e}} < 1$, and

$$\lim_{n\longrightarrow\infty} \frac{n\mathrm{e}^{-n}}{\mathrm{e}^{-n/2}} = \lim_{n\longrightarrow\infty} n\mathrm{e}^{-n/2} = 0,$$

since the exponential dominates. Thus, our series also converges.

The *Comparison Test* can be used as well. If $f(x) = x\mathrm{e}^{-x/2}$,

$$f'(x) = x'\mathrm{e}^{-x/2} + x\big(\mathrm{e}^{-x/2}\big)' = \mathrm{e}^{-x/2} + x\mathrm{e}^{-x/2}\cdot(-1/2) = \frac{1}{2}\mathrm{e}^{-x/2}(2-x).$$

So $f(x) \le f(2) = 2\mathrm{e}^{-2/2} < 1$ for $x \ge 2$ and thus $n\mathrm{e}^{-n} \le \mathrm{e}^{-n/2}$ for all n. Since $n\mathrm{e}^{-n} \ge 0$ and $\sum_{n=1}^{\infty} \mathrm{e}^{-n/2}$ converges being a geometric series with $r = 1/\sqrt{\mathrm{e}} < 1$, our series also converges.

The *Integral Test* can also be used. The function $f(x) = x\mathrm{e}^{-x}$ is positive for $x \ge 1$. Since

$$f'(x) = x'\mathrm{e}^{-x} + x\big(\mathrm{e}^{-x}\big)' = \mathrm{e}^{-x} + x\mathrm{e}^{-x}\cdot(-1) = \mathrm{e}^{-x}(1-x),$$

$f(x)$ is decreasing for $x \ge 1$. So the sum converges if and only if $\int_1^{\infty} x\mathrm{e}^{-x}\mathrm{d}x$ does. Integration by parts gives

$$\begin{aligned}\int_1^{\infty} x\mathrm{e}^{-x}\mathrm{d}x &= -\int_1^{\infty} x\mathrm{d}\mathrm{e}^{-x} = -\left(x\mathrm{e}^{-x}\Big|_1^{\infty} - \int_1^{\infty}\mathrm{e}^{-x}\mathrm{d}x\right) = -\left(\lim_{x\longrightarrow\infty} x\mathrm{e}^{-x} - 1\mathrm{e}^{-1} + \mathrm{e}^{-x}\Big|_1^{\infty}\right)\\ &= -\big(0 - \mathrm{e}^{-1} + \mathrm{e}^{-\infty} - \mathrm{e}^{-1}\big) = 2\mathrm{e}^{-1}.\end{aligned}$$

Since the integral is finite, $\sum_{n=1}^{\infty} n\mathrm{e}^{-n}$ converges.

(b; 8pts) *Find all values of p for which the following series converges*

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

Write your answer in the box to the right and justify it below.

$$\boxed{p > 1}$$

If $p \le 0$, $1/(n(\ln n)^p) \ge 1/n$ whenever $\ln n \ge 1$ (so for $n \ge 3$). Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-series test and $0 \le 1/(n(\ln n)^p)$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges by the *Comparison Test* whenever $p \le 0$.

Suppose $p > 0$. The function $f(x) = 1/(x(\ln x)^p)$ is then positive and decreasing for $x \ge 2$. Thus, the sum converges if and only if

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} \mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u^p} \mathrm{d}u$$

does. For $p = 1$, we get

$$\int_{\ln 2}^{\infty} \frac{1}{u} \mathrm{d}u = \ln u \Big|_{\ln 2}^{\infty} = \ln \infty - \ln \ln 2 = \infty;$$

so the sum diverges. If $p < 1$,

$$\int_{\ln 2}^{\infty} \frac{1}{u^p} \mathrm{d}u = \frac{1}{1-p} u^{1-p} \Big|_{\ln 2}^{\infty} = \frac{1}{1-p} \big(\infty^{1-p} - (\ln 2)^{1-p}\big) = \infty\,,$$

since $1-p > 0$; so the sum diverges. Finally, if $p > 1$,

$$\begin{aligned}\int_{\ln 2}^{\infty} \frac{1}{u^p} \mathrm{d}u = \frac{1}{-p+1} u^{-p+1} \Big|_{\ln 2}^{\infty} = -\frac{1}{p-1} \big(\infty^{-(p-1)} - (\ln 2)^{-(p-1)}\big) &= -\frac{1}{p-1} \big(0 - (\ln 2)^{-(p-1)}\big) \\ &= \frac{1}{p-1} (\ln 2)^{-(p-1)}\,,\end{aligned}$$

since $p-1 > 0$; so the sum converges. Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$. This is the same answer as in the p-Series Test.

Problem 6 (20pts)

For each of the following series, determine whether it converges and if so, find its sum. Simplify your answers as much as possible and justify them.

(a; 8pts) $\displaystyle\sum_{n=1}^{\infty}\frac{1+4^n}{1+3^n}$

Since $4^n \ge 3^n$, $\dfrac{1+4^n}{1+3^n} \ge 1$. Since $1 \ge 0$ and the series

$$\sum_{n=1}^{\infty} 1 = 1+1+1+\dots$$

diverges, our series $\boxed{\text{diverges}}$ by the *Comparison Test.*

We can also use the *Test for Divergence*:

$$\lim_{n\longrightarrow\infty}\frac{1+4^n}{1+3^n} = \lim_{n\longrightarrow\infty}\frac{1/3^n+4^n/3^n}{1/3^n+3^n/3^n} = \lim_{n\longrightarrow\infty}\frac{1/3^n+(4/3)^n}{1/3^n+1} = \frac{0+(4/3)^\infty}{0+1} = \infty.$$

Since the limit of the initial sequence is not zero (it does not even exist by MAT 127 definition), the series $\boxed{\text{diverges}}$

We can also compare to the divergent geometric series

$$\sum\frac{1}{2}\cdot\frac{4^n}{3^n} = \frac{1}{2}\sum\left(\frac{4}{3}\right)^n,$$

as both series have positive terms. Since

$$\begin{aligned}\frac{1+4^n}{1+3^n}-\frac{1}{2}\left(\frac{4}{3}\right)^n &= \frac{1+4^n}{1+3^n}-\frac{1}{2}\cdot\frac{4^n}{3^n} = \frac{1}{2}\cdot\frac{2\cdot3^n(1+4^n)-4^n(1+3^n)}{(1+3^n)3^n}\\ &= \frac{3^n\cdot4^n-4^n+2\cdot3^n}{2\cdot3^n(1+3^n)} = \frac{4^n(3^n-1)+2\cdot3^n}{2\cdot3^n(1+3^n)} \ge 0,\end{aligned}$$

our series has bigger terms and therefore must also $\boxed{\text{diverge}}$

Alternatively, since the terms in our series *look like* $4^n/3^n = (4/3)^n$, we can *limit-compare* this series to the geometric series $\sum(4/3)^n$; this diverges, since $|4/3| \ge 1$. This limit-comparison can be made, since both series have positive terms and

$$\begin{aligned}\frac{(1+4^n)/(1+3^n)}{4^n/3^n} &= \frac{1+4^n}{4^n}\cdot\frac{3^n}{1+3^n} = \big(1/4^n+1\big)\cdot\frac{3^n/3^n}{1/3^n+3^n/3^n}\\ &= \big(1/4^n+1\big)\cdot\frac{1}{1/3^n+1} \longrightarrow (0+1)\cdot\frac{1}{0+1} = 1.\end{aligned}$$

Since the limit is nonzero, our series $\boxed{\text{diverges}}$ because the other series does.

Finally, we can also use the *Ratio Test* for Series:

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} = \frac{(1+4^{n+1})/(1+3^{n+1})}{(1+4^n)/(1+3^n)} = \frac{1+4^{n+1}}{1+4^n}\cdot\frac{1+3^n}{1+3^{n+1}} &= \frac{1/4^n+4^{n+1}/4^n}{1/4^n+4^n/4^n}\cdot\frac{1/3^n+3^n/3^n}{1/3^n+3^{n+1}/3^n}\\ &= \frac{1/4^n+4}{1/4^n+1}\cdot\frac{1/3^n+1}{1/3^n+3} \longrightarrow \frac{0+4}{0+1}\cdot\frac{0+1}{0+3} = \frac{4}{3}.\end{aligned}$$

Since $4/3 > 1$, the series $\boxed{\text{diverges}}$

(b; 12pts) $\displaystyle\sum_{n=2}^{\infty}\frac{2}{n^2-1}$ *Hint:* partial fractions.

By (quick) partial fractions,

$$\frac{1}{n^2-1}=\frac{1}{(n-1)(n+1)}=\frac{1}{+1-(-1)}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)=\frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

Thus, the sequence of partial sums is given by

$$\begin{aligned}
s_n &= \sum_{k=2}^{k=n}\left(\frac{1}{k-1}-\frac{1}{k+1}\right)\\
&= \left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)\\
&= 1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}
\end{aligned}$$

The second equality above is obtained by canceling the negative term in the k-th summand for $k=2,3,\ldots,n-2$ with the positive term two summands later. This leaves the positive terms in the first two summands and the negative terms in the last two summands (this also gives the right answer for $n=2$). Since $1/n \longrightarrow 0$ as $n\longrightarrow\infty$, the sequence s_n converges; thus the series also converges and

$$\sum_{n=1}^{\infty}\frac{2}{n^2-1}=\lim_{n\longrightarrow\infty}s_n=1+\frac{1}{2}=\boxed{\frac{3}{2}}$$

Below are alternative ways to show that the series *converges*, but they do not determine its limit.

Since the terms in the above series look like $\sum 1/n^2$, limit-compare the given series to $\sum 1/n^2$. The latter series is a p-series with $p=2>1$ and thus converges. This limit-comparison is applicable here because both series have positive terms and

$$\frac{1/(n^2-1)}{1/n^2}=\frac{1}{(n^2-1)/n^2}=\frac{1}{1-1/n^2}\longrightarrow\frac{1}{1-1/\infty}=1.$$

Thus, our series converges because the other series does.

We can also compare our series to the convergent p-series

$$\sum\frac{1}{n^2/2}=2\sum\frac{1}{n^2}.$$

Since $n^2-1>n^2/2$ if $n\ge 2$, $1/(n^2-1)<1/(2n^2)$. Since both series have positive terms and the "larger" p-series converges, our series also converges.

The Integral Test can also be applied with $f(x)=1/(x^2-1)$, since this function is positive and decreasing for $x\ge 2$. In order to compute the integral, use the partial fractions above and carefully take the limit of anti-derivative as $x\longrightarrow\infty$.

Grading for 6a: *sum converges* 0pts regardless of explanation; *sum diverges* 1pt; justification up to 7pts, with penalties for missing condition checks (depending on convergence/divergence test used) and computational errors

Grading for 6b: *PFs* correct 4pts (-1pt each if 1/2 is wrong or the overall sign is reversed; -2pts if fractions have the same sign; -2pts for wrong denominators); clear indication or statement of two-step cancellation 3pts; simplifying to final answer for s_n 1pt; limit of $\{s_n\}$ and justification 1pt each; *series converges* and sum 1pt each; no penalty for carryover errors if feasible (e.g. if 1/2 is incorrect; if PFs are badly messed up, resulting in no cancellations in s_n, likely loss of all subsequent points); if at any point, the infinite sum is split into two divergent sums, no more than 8pts for the question even if the rest is done right.

Grading for 6b (non-PF approach): *sum diverges* 0pts regardless of explanation; *sum converges* 2pts; justification up to 3pts (including positivity statements); *not* in addition to any points above