

## Rigidity in the positive mass theorem with charge

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The article by Thorne Lay and Hiroo Kanamori is an excellent review of the relationship between seismic moment and energy release. However, they would find that the relationship between seismic moment and energy release is not linear. A 100-megaton nuclear explosion releases five times as much energy as a 20-megaton nuclear explosion.

while that of a 100-megaton nuclear explosion is approximately five times as much energy as a 20-megaton nuclear detonation event—a 40-megaton atmospheric nuclear device had still more energy by a factor of about 3, or 15 times the energy of a 10-megaton nuclear device. I believe the authors used the relation for seismic energy release rather than total strain energy release. The seismic energy underestimates the total strain energy release by a variable that depends on the fault plane. Accounting for total strain energy release would increase the earthquake energy number by orders of magnitude.

Despite the catastrophic damage potential of nuclear bombs, the forces of nature occasionally unleash much larger energy releases. Although the nuclear bombs are under our control, earthquakes, volcanic eruptions, and extreme weather events are not. However, by judicious preparation and avoidance measures, humans can significantly diminish the damage of natural events.

This article does not have any references.

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By the act of hitting a ball with a bat, one calculates the force energy to deliver the ball to its new location, but one must also take into account that the ball extended its energy release to that location which became struck by the ball as its momentum ceased and passed energy to the struck team. Therefore the parameters of the damage extend into the future when the received energy to that pushed upon, later becomes released in a new event. Perhaps calculations of one added that in while another's calculations did not. E.M.C.  
Written by Edgar McCarvill, 14 July 2012 19:59



## Rigidity in the positive mass theorem with charge

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In this paper, we show how a natural coupling of the Dirac equation with the generalized Jang equation leads to a proof of the rigidity statement in the positive mass theorem with charge, without the maximal slicing condition, provided a solution to the coupled system exists. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4820469>]

### I. INTRODUCTION

Consider an initial data set for the Einstein equations  $(M, g, k)$ , where  $M$  is a 3-manifold,  $g$  a Riemannian metric, and  $k$  a symmetric 2-tensor. The energy and momentum densities are given by

$$2\mu = R + (\text{Tr } k)^2 - |k|^2,$$

$$J = \text{div}(k - (\text{Tr } k)g),$$

and the dominant energy condition is

$$\mu \geq |J|. \tag{1}$$

For simplicity, we will throughout assume that the data set possesses a single strongly asymptotically flat end, that is, a region of  $M$  which is diffeomorphic to the complement of a ball in  $\mathbb{R}^3$ , with the property that the metric  $g$  and tensor  $k$  expressed in coordinates induced by this diffeomorphism decay with the following rates:

$$g_{ij} - \delta_{ij} = O_2(r^{-1}), \quad k_{ij} = o_1(r^{-2}),$$

where  $r$  is the Euclidean distance in these asymptotically flat coordinates, and where  $O_p(r^{-\beta})$  ( $o_p(r^{-\beta})$ , respectively) means a function  $f$  such that  $\sum_{j=0}^p r^{\beta+j} |D^j f|$  is bounded (tends to zero respectively) as  $r \rightarrow \infty$ . We will say that the initial data set is strongly asymptotically flat if it is the union of a compact set and a strongly asymptotically flat end. The total mass of the end is defined by

$$m = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} (g_{ij,j} - g_{jj,i}) v^i,$$

where  $S_r$  is the coordinate sphere of radius  $r$  and  $v$  is its unit outward normal.

The Riemannian manifold  $(M, g)$  is assumed to be complete with boundary, where the boundary  $\partial M$  is an outermost apparent horizon. By an apparent horizon, we mean that  $\partial M$  is compact and each component has mean curvature  $H_{\partial M} = \pm \text{Tr}_{\partial M} k$  where the trace  $\text{Tr}_{\partial M}$  is taken with respect to the induced metric on  $\partial M$ . The apparent horizon is outermost if it is not “contained” in any other apparent horizon. To make sense of the term “contain,” we restrict our attention to surfaces which bound a 3-dimensional region containing the boundary. Recall the positive mass theorem.

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**Theorem 1.** *Let  $(M, g, k)$  be initial data with a strongly asymptotically flat end and satisfying the dominant energy condition. Then*

$$m \geq 0,$$

*and equality holds if and only if  $(M, g, k)$  arises from Minkowski space.*

The case of equality in this theorem is referred to as the rigidity statement. Note that our hypothesis of strong asymptotic flatness implies the vanishing of the ADM linear momentum. Under weaker hypotheses, in which the linear momentum is possibly non-zero, the rigidity statement was proved in Refs. 1 and 2.

Two entirely different approaches to prove this theorem are relevant to our work in this paper. The first one, due to Schoen and Yau was given in two installments, first in the maximal slice case, that is when  $\text{Tr} k = 0$ , using minimal surface theory.<sup>3</sup> This result was then used to prove the general case by applying the Jang equation.<sup>4</sup> The main idea in this second step is to look for a hypersurface  $\bar{M}$ , in the Riemannian product  $(M \times \mathbb{R}, g + dt^2)$ , given as the graph of a function  $f$  on  $M$ , and such that its mean curvature satisfies  $H_{\bar{M}} = \text{Tr}_{\bar{M}} k$ . This last equation is known as the Jang equation, and its validity ensures that the the scalar curvature of  $\bar{g} = g + df^2$  (the induced metric on  $\bar{M}$ ), is in some sense weakly non-negative, so that there is a conformal metric  $\tilde{g} = u^4 \bar{g}$  with zero scalar curvature. One may then apply the maximal slice result to obtain the desired conclusion.

A second approach is due to Witten and is based on solutions of the Dirac equation.<sup>5</sup> Here, using a harmonic spinor  $\psi$ , which tends to a constant spinor  $\psi_0$  at infinity with  $|\psi_0| \rightarrow 1$ , one derives an integral identity which exhibits the mass explicitly as the integral of a non-negative quantity over  $M$ .<sup>6</sup>

Now consider the generalization of Theorem 1 to initial data sets for the Einstein-Maxwell equations. For simplicity, we restrict ourselves to the case where the initial magnetic field vanishes. Thus, we consider initial data sets  $(M, g, k, E)$  where  $E$  is a vector field on  $M$ . We say that this data set is strongly asymptotically flat if in addition to the decay condition above we also have

$$E = O_1(r^{-2}).$$

The energy density of the matter content after the contribution of the electric field has been removed is given by

$$2\mu_0 = R + (\text{Tr} k)^2 - |k|^2 - 2|E|^2.$$

Define half the charge density by

$$\rho = \frac{1}{2} \text{div} E,$$

and denote the total charge

$$Q = \lim_{r \rightarrow \infty} \frac{1}{4\pi} \int_{S_r} g(E, \nu).$$

The charged dominant energy condition, in Gaussian units with  $c = G = 1$ , then takes the form

$$\mu_0 \geq |J| + |\rho|. \quad (2)$$

For the rigidity statement in the positive mass theorem to follow, the role of Minkowski space is replaced by the Majumdar-Papapetrou spacetime, which we will denote by MP. This spacetime has the manifold structure  $\mathbb{R} \times (\mathbb{R}^3 \setminus \{p_1, \dots, p_l\})$  and is equipped with the metric

$$g_{\text{MP}} = \phi_{\text{MP}}^2 dt^2 + \phi_{\text{MP}}^{-2} \delta,$$

where  $p_1, \dots, p_l$  are  $l$  points in  $\mathbb{R}^3$ ,

$$\phi_{\text{MP}} = \left( 1 + \sum_{i=1}^l \frac{m_i}{r_i} \right)^{-1}, \quad (3)$$

$\delta$  is the flat metric on  $\mathbb{R}^3$ ,  $r_i$  is the Euclidean distance to  $p_i$ , and the constants  $m_i$  are all non-negative. With the electric field  $E_{\text{MP}} = \nabla \log \phi_{\text{MP}}$ , this spacetime can be interpreted as  $l$  black holes in equilibrium, where the  $i$ th black hole has mass  $m_i$  and charge  $m_i$ . In other words, the Lorentzian spacetime  $(\mathbb{R} \times (\mathbb{R}^3 \setminus \{p_1, \dots, p_l\}), g_{\text{MP}})$  is static and electrovacuum, and asymptotically flat except for  $l$  cylindrical ends.

The first part of the following theorem was proved in Ref. 7 (see also Ref. 8), using a modification of Witten's approach. The rigidity statement was established in Ref. 9.

**Theorem 2.** *Let  $(M, g, k, E)$  be initial data for the Einstein-Maxwell equations with a strongly asymptotically flat end, and satisfying the charged dominant energy condition, then*

$$m \geq |Q|.$$

*Suppose in addition that  $\text{Tr} k = 0$ , then equality holds if and only if the data set  $(M, g, k, E)$  arises from the Majumdar-Papapetrou spacetime.*

The purpose of the present paper is to remove the maximal slice hypothesis in the above theorem, by coupling the Dirac equation with a natural modification of the Jang equation. This latter equation was introduced by Bray and Khuri<sup>10,11</sup> in an attempt to reduce the general Penrose inequality to the Riemannian Penrose inequality. This was carried out successfully in the spherically symmetric case. In this approach, one still looks for a hypersurface  $\bar{M}$  satisfying an equation of the form  $H_{\bar{M}} = \text{Tr}_{\bar{M}} K$  (where  $K$  is an extension of  $k$ ) but in a warped product metric  $g + \phi^2 dt^2$  rather than in the product metric. This is of course better adapted to the task at hand, since the MP metric is itself a warped product metric.

In Sec. II, we formulate the coupled Dirac-Jang system of equations. In Sec. III, it is shown that if this system has a solution, then the rigidity statement of Theorem 2 follows without the maximal slice hypothesis. Finally in Sec. IV, we present evidence which indicates that the system of Sec. II can be solved. Some of the most technical computations are relegated to the Appendix.

## II. THE COUPLED SYSTEM

### A. The generalized Jang equation

Let  $\phi : M \rightarrow \mathbb{R}$  be a positive function, and consider the warped product metric  $g + \phi^2 dt^2$  on  $M \times \mathbb{R}$ . Let  $f : M \rightarrow \mathbb{R}$  be a smooth function, and denote its graph by  $\bar{M} = \{(x, f(x)) : x \in M\} \subset M \times \mathbb{R}$ . The induced metric on  $\bar{M}$  arising from the warped product metric is given by  $\bar{g} = g + \phi^2 df^2$ . The generalized Jang equation is then given by  $H_{\bar{M}} = \text{Tr}_{\bar{M}} K$ , where  $H_{\bar{M}}$  is the mean curvature of the graph and  $K$  is a particular extension of the initial data  $k$  to the 4-manifold  $M \times \mathbb{R}$  (see Ref. 10). In local coordinates, this equation becomes

$$\left( g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2 |\nabla f|^2} \right) \left( \frac{\phi \nabla_{ij} f + \phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2 |\nabla f|^2}} - k_{ij} \right) = 0, \quad (4)$$

where  $\nabla_{ij} f$  are covariant derivatives with respect to  $g$  and  $f^i = g^{ij} f_j$ . The purpose of the generalized Jang equation is to give positivity properties to the scalar curvature of  $\bar{g}$ ; this will be discussed further below. The basic existence theory for this equation has been established in Ref. 12.

### B. The Dirac equation

Dirac spinors are cross sections of a vector bundle  $\mathcal{S}$  associated to a principal bundle over  $\bar{M}$  with structure group  $SL(2, \mathbb{C})$ , the simply connected double cover of  $SO(3, 1)$ , see Ref. 13 for details. We write a Dirac equation on  $\bar{M}$  which will then couple with the generalized Jang equation. The vector bundle  $\mathcal{S}$  is equipped with a connection compatible with the Jang metric  $\bar{g}$  as follows:

$$\bar{\nabla}_{e_i} = e_i + \frac{1}{4} \bar{\omega}_{ijl} c(e^j) c(e^l),$$

where  $\bar{\omega}_{ijl}$  are the affine connection coefficients of the Jang metric  $\bar{g}$  defined by

$$\bar{g}(\bar{\nabla}_{e_i} e_j, e_l) = \bar{\omega}_{ijl},$$

$e_1, e_2, e_3$  is an orthonormal frame on  $\bar{M}$ , and  $c : T^*\bar{M} \rightarrow \text{End}(S)$  is global Clifford multiplication. Consider the Einstein-Maxwell spin connection on  $S$

$$\nabla_{e_i} = \bar{\nabla}_{e_i} - \frac{1}{2} c(\bar{E}) c(e_i) c(e_0), \quad (5)$$

where  $\bar{E}$  is the electric field associated with the Jang surface to be defined below, and  $e_0$  is the unit normal to  $\bar{M}$ . An important observation is that because of the electric field contribution this connection is *not* metric compatible.

In order to define  $\bar{E}$ , let  $F = F_{ab} dx^a \wedge dx^b$  be the field strength tensor on  $M \times \mathbb{R}$ , given by  $F_{0i} = \phi E_i$  and  $F_{ij} = 0$  for  $i, j = 1, 2, 3$ , where  $x^0 = t$ , and the remaining  $x^i$ ,  $i = 1, 2, 3$ , are local coordinates on  $M$ . Then set

$$\bar{E}_i = F(N, X_i) = \frac{E_i + \phi^2 f_i f^j E_j}{\sqrt{1 + \phi^2 |\nabla f|_g^2}},$$

where  $X_i = \partial_i + f_i \partial_t$ ,  $i = 1, 2, 3$  are basis elements for the tangent space to  $\bar{M}$  and

$$N = \frac{f^i \partial_i - \phi^{-2} \partial_t}{\sqrt{\phi^{-2} + |\nabla f|^2}}$$

is the unit normal to  $\bar{M}$ . This induced electric field on the Jang surface first appeared in Ref. 14, and has special properties to be exploited below.

Let  $\Gamma(S)$  be the space of cross-sections of the bundle  $S$ . The Einstein-Maxwell Dirac operator  $\mathcal{D} : \Gamma(S) \rightarrow \Gamma(S)$  is now defined as usual

$$\mathcal{D}\psi = \sum_{i=1}^3 c(e_i) \nabla_{e_i} \psi. \quad (6)$$

A spinor  $\psi$  on  $\bar{M}$  is a harmonic spinor if it satisfies the Dirac equation

$$\mathcal{D}\psi = 0. \quad (7)$$

### C. The Dirac-Jang system

We can now formulate the coupled system appropriate for our needs, by choosing the warping factor  $\phi$  as follows.

**Definition 3.** Let  $f : M \rightarrow \mathbb{R}$  with  $f \in C^{2,\beta}(M)$ , and  $\psi \in \Gamma(S)$  with  $\psi \in C^{1,\beta}(M)$ . We say that  $(f, \psi)$  is a solution of the Dirac-Jang system if  $f$  satisfies (4),  $\psi$  satisfies (7), and

$$\phi = |\psi|^2. \quad (8)$$

Let us now investigate the appropriate asymptotics for solutions to the coupled Dirac-Jang system. First, the asymptotics at spatial infinity are standard and are given by

$$f = O_2(|x|^{-\frac{1}{2}}) \quad \text{and} \quad \psi \rightarrow \psi_0 \quad \text{as} \quad |x| \rightarrow \infty, \quad (9)$$

where  $\psi_0$  is a fixed constant spinor at infinity with  $|\psi_0| = 1$ . Note that this fall-off for  $f$  guarantees that the mass of the Jang metric  $\bar{g}$  agrees with that of  $g$ , and that the total charge of  $\bar{E}$  agrees with that of  $E$ .

In order to motivate the asymptotics at the apparent horizon, consider the MP spacetime with one black hole (extreme Reissner-Nordström spacetime) with metric

$$-\left(1 - \frac{m}{r}\right)^2 dt^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2 d\sigma^2.$$

Let  $t = f(r)$  be a radial graph, with induced metric

$$g = \left( \left(1 - \frac{m}{r}\right)^{-2} - \left(1 - \frac{m}{r}\right)^2 f'^2 \right) dr^2 + r^2 d\sigma^2.$$

As is calculated in Ref. 11, the second fundamental form of the graph is given by

$$k_{ij} = \frac{\phi \nabla_{ij} f + \phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2 |\nabla f|^2}},$$

where  $\phi = 1 - \frac{m}{r}$  and the covariant derivatives are calculated with respect to the metric  $g$ . Thus  $(M = \mathbb{R}^3 - B_m(0), g, k)$  forms an initial data set for which the graph  $t = f(r)$  is a solution of the generalized Jang equation. If  $f$  grows faster than  $(r - m)^{-1}$  then  $g$  will not be a Riemannian metric, that is, the graph will not be spacelike. Moreover, if  $f$  grows slower than  $(r - m)^{-1}$  then  $M$  will not have a boundary, but will rather have a cylindrical end as in the MP initial data. Thus, we will assume that

$$f(r) \sim \int_r^{r_0} \left(1 - \frac{m}{r}\right)^{-2} \sim (r - m)^{-1},$$

where  $\sim$  indicates the presence of other added lower order terms. In order to see that the boundary is in fact a future apparent horizon with these asymptotics for  $f$ , let us note that

$$g_{11} = \left(1 - \frac{m}{r}\right)^{-2} - \left(1 - \frac{m}{r}\right)^2 f'^2 \sim O(1).$$

Therefore, the distance to the boundary is given by

$$\tau = \int_m^r \sqrt{g_{11}} \sim r - m.$$

We now compute the future null expansion of the coordinate spheres with respect to the initial data metric  $g$ . A standard formula yields the mean curvature of coordinate spheres

$$H_{S_r} = \frac{2\sqrt{g^{11}}}{r},$$

and the trace of the initial data  $k$  over the coordinate spheres is given by

$$\text{Tr}_{S_r} k = -\frac{\phi \gamma^{ij} \Gamma_{ij}^1 f'}{\sqrt{1 + \phi^2 |\nabla f|^2}} = \frac{2}{r} \frac{\phi g^{11} f'}{\sqrt{1 + \phi^2 g^{11} f'^2}},$$

where  $\gamma_{ij}$  is the induced metric on  $S_r$  and  $\Gamma_{ij}^l$  are the Christoffel symbols of  $g$ . It follows that the future null expansion becomes

$$\begin{aligned} \theta_+ &= H_{S_r} + \text{Tr}_{S_r} k = \frac{2}{r} \left( \sqrt{g^{11}} + \frac{\phi g^{11} f'}{\sqrt{1 + \phi^2 g^{11} f'^2}} \right) \\ &= \frac{2}{r} \left( \frac{g^{11}}{1 + \phi^2 g^{11} f'^2} \right) \left( \sqrt{g^{11}} - \frac{\phi g^{11} f'}{\sqrt{1 + \phi^2 g^{11} f'^2}} \right)^{-1} \\ &\sim (r - m)^2 \\ &\sim \tau^2, \end{aligned}$$

so that  $\partial M$  is a future apparent horizon.

In conclusion, we expect that solutions of the Dirac-Jang system will follow similar asymptotics

$$f \sim \tau^{-1} \quad \text{and} \quad \psi \sim \tau^{\frac{1}{2}} \quad \text{as} \quad x \rightarrow \partial M. \quad (10)$$

Notice also that with these asymptotics, the Jang metric still possesses an infinitely long cylindrical neck since

$$\bar{g} = g + \phi^2 df^2 \sim O(1) + \tau^2 \tau^{-4} \sim \tau^{-2},$$

and so

$$\bar{\tau} = \int_0^\tau \sqrt{\bar{g}_{11}} \sim -\log \tau,$$

where  $\bar{\tau}(x) = \text{dist}_{\bar{g}}(x, \partial M)$ . Therefore, the asymptotics lead to behavior for the Jang metric which is similar to that of initial data in the MP spacetime.

### III. THE POSITIVE MASS THEOREM WITH CHARGE

In this section, we prove the following theorem.

**Theorem 4.** *Let  $(M, g, k, E)$  be strongly asymptotically flat initial data for the Einstein-Maxwell equations satisfying the charged dominant energy condition, with total mass  $m$  and total charge  $Q$ , and suppose that the Dirac-Jang system has a solution  $(f, \psi)$  which satisfies (9) and (10). Then*

$$m \geq |Q|,$$

and equality holds if and only if  $(M, g, k, E)$  arises from the Majumdar-Papapetrou spacetime.

Note that the  $t = 0$  slice of the Majumdar-Papapetrou spacetime does not fall under the hypotheses of this theorem, as it possesses cylindrical ends. This is related to the fact that we employ blow-up solutions of the generalized Jang equation at the apparent horizon boundary. In order to treat initial data containing cylindrical ends, more general boundary conditions for the generalized Jang equation should be utilized.

The starting point of the proof is the Lichnerowicz identity, which now takes the form

$$\not{D}^2 = \nabla^* \nabla + \frac{1}{4} \mathcal{R}, \tag{11}$$

where  $\mathcal{R}$  is the following endomorphism of  $\mathcal{S}$

$$\mathcal{R} = \bar{R} - 2|\bar{E}|^2 - 2\bar{\rho} c(e_0),$$

with  $\bar{R}$  the scalar curvature of the Jang metric  $\bar{g}$  and  $\bar{\rho} = \frac{1}{2} \bar{\text{div}} \bar{E}$  the charge density on  $\bar{M}$ . Applying (11) to the harmonic spinor  $\psi$ , taking the inner product with  $\psi$ , and integrating by parts over  $\bar{M}$  produces

$$\int_{\bar{M}} |\nabla \psi|^2 + \frac{1}{4} \langle \mathcal{R} \psi, \psi \rangle = 4\pi(m - |Q|). \tag{12}$$

The right-hand side is the boundary term at spatial infinity, and no interior boundary terms appear in light of the asymptotics (10).

We will now show that the second term on the left-hand side of (12) is non-negative. As proved in Ref. 14, our choice for the electric field  $\bar{E}$  on the Jang surface ensures that

$$|E| \geq |\bar{E}|, \tag{13}$$

and

$$\rho = \bar{\rho} \sqrt{1 + \phi^2 |\nabla f|^2}. \tag{14}$$

Moreover, due to the fall off rate of the Jang graph the total charge is unchanged

$$\lim_{r \rightarrow \infty} \frac{1}{4\pi} \int_{S_r} \bar{g}(\bar{E}, \bar{\nu}) = Q. \tag{15}$$

In Ref. 10, it was shown that the generalized Jang equation implies the following formula for the scalar curvature of  $\bar{g}$

$$\bar{R} = 2(\mu - J(w)) + |h - K|^2 + 2|q|^2 - 2\phi^{-1} \bar{\text{div}}(\phi q), \tag{16}$$

where  $h$  is the second fundamental form of the Jang surface  $\bar{M}$ ,  $K$  is a specially chosen extension of the initial data  $k$ , and

$$w^i = \frac{\phi f^i}{\sqrt{1 + \phi^2 |\nabla f|^2}}, \tag{17}$$

$$q_i = w^j (h_{ij} - K_{ij}). \tag{18}$$

Notice that (2), (13), and (16), together with  $|w| \leq 1$  imply

$$\begin{aligned} \bar{R} - 2|\bar{E}|^2 &= 2(\mu_0 - J(w)) + 2(|E|^2 - |\bar{E}|^2) + |h - K|^2 + 2|q|^2 - 2\phi^{-1} \overline{\text{div}}(\phi q) \\ &\geq 2|\rho| - 2\phi^{-1} \overline{\text{div}}(\phi q). \end{aligned} \tag{19}$$

We conclude, in view of the coupling  $\phi = |\psi|^2$ , and  $|\rho| \geq |\bar{\rho}|$ , that the second term on the left-hand side in (12) satisfies

$$\frac{1}{4} \langle \mathcal{R}\psi, \psi \rangle \geq \frac{1}{2} (|\rho| |\psi|^2 - \langle \bar{\rho} c(e_0)\psi, \psi \rangle - \overline{\text{div}}(\phi q)) \geq -\frac{1}{2} \overline{\text{div}}(\phi q). \tag{20}$$

Thus, (12) becomes

$$\int_{\bar{M}} |\nabla \psi|^2 = \int_{\bar{M}} |\nabla \psi|^2 - \frac{1}{2} \overline{\text{div}}(\phi q) \leq 4\pi(m - |Q|). \tag{21}$$

Note that the integral of the divergence term vanishes according to the asymptotics (9) and (10).

Now consider the case of equality in (21). It follows that the harmonic spinor  $\psi$  is covariantly constant  $\nabla \psi = 0$ . We now recall that  $\nabla$  is not metric compatible, so that although  $\psi$  is parallel,  $|\psi|$  is not constant. In fact, we show in the Appendix that  $\phi = |\psi|^2$  satisfies the elliptic equation

$$\bar{\Delta} \phi - \frac{1}{2} \bar{R} \phi = 0, \tag{22}$$

where  $\bar{\Delta}$  is the Laplace-Beltrami operator with respect to  $\bar{g}$ . Furthermore, it now follows from the energy condition (2), as well as (19) and (20), that at each point of  $M$  either

$$\mu_0 = |J| = \rho = \bar{\rho} = |E| - |\bar{E}| = |h - K| = |q| = 0, \tag{23}$$

or  $|\psi| = 0$ . Since

$$\phi(x) \rightarrow 0 \text{ as } x \rightarrow \partial M, \text{ and } \phi(x) \rightarrow 1 \text{ as } x \rightarrow \infty, \tag{24}$$

we may apply the Hopf maximum principle to conclude that  $\phi > 0$  on the interior of  $M$ . It then follows that in fact (23) holds without any preconditions. In particular

$$\bar{R} = 2|\bar{E}|^2. \tag{25}$$

In conclusion, in view of (25) and  $\mu_0 = |J| = \bar{\rho} = 0$ , we find that  $(\bar{M}, \bar{g}, \bar{E})$  is a time symmetric electrovacuum (and asymptotically flat) initial data set, with

$$m = |Q|.$$

According to Chrusciel *et al.*,<sup>9</sup> the only such electrovacuum initial data are the MP initial data, and hence  $\bar{g} = g_{\text{MP}}$  and  $\bar{E} = E_{\text{MP}}$ . We now have  $g = g_{\text{MP}} - \phi^2 df^2$ , so that the map  $x \mapsto (x, f(x))$  yields an isometric embedding of  $(M, g)$  into the spacetime  $(\mathbb{R} \times M, -\phi^2 dt^2 + g_{\text{MP}})$ . Next observe that since  $\phi_{\text{MP}}$  satisfies the same Eq. (22) and boundary conditions (24) as  $\phi$ , we must have  $\phi = \phi_{\text{MP}}$ . Therefore,  $(\mathbb{R} \times M, -\phi^2 dt^2 + g_{\text{MP}})$  is the MP spacetime.

It remains to show that  $k$  and  $E$  are respectively the second fundamental form of and induced electric field on the isometric embedding  $(M, g) \hookrightarrow \text{MP}$ . This, however, follows from previous work. Namely, since  $|h - K| = 0$  it is shown in Refs. 10 and 11 that  $k$  is the desired second fundamental form, and the fact that  $E$  is the induced electric field on the embedding is shown in Ref. 14.

This completes the proof of Theorem 4.

#### IV. A CONJECTURE AND SOME EVIDENCE

In this section, we present evidence for the following conjecture.

**Conjecture 1.** *Let  $(M, g, k, E)$  be a strongly asymptotically flat initial data set for the Einstein-Maxwell equations. Then the Dirac-Jang system has a solution  $(f, \psi)$  which satisfies (9) and (10).*

We note that if proved, this statement, in conjunction with Theorem 4, would also give a proof of the following conjecture.

**Conjecture 2.** *Let  $(M, g, k, E)$  be a strongly asymptotically flat initial data set for the Einstein-Maxwell equations satisfying the charged dominant energy condition, with total mass  $m$  and total charge  $Q$ . Then*

$$m \geq |Q|,$$

*and equality holds if and only if  $(M, g, k, E)$  arises from the Majumdar-Papapetrou spacetime.*

Observe that the Dirac-Jang system is truly a coupled system of equations, since the metric appearing in the Dirac equation depends on the solution to the generalized Jang equation. However, both are elliptic, or degenerate elliptic in the case of the generalized Jang equation (the degeneracy appears only at the horizons), and so we have the Schauder estimates available to analyze (heuristically) whether it is possible to solve this set of equations. Ultimately we would like to apply a standard iteration procedure to obtain existence. Namely, pick an arbitrary positive function  $\phi_0$  with the correct boundary conditions, and then use it to solve the generalized Jang equation for  $f_0$ . Use  $\phi_0$  and  $f_0$  to construct  $\bar{g}_0$ , and then solve the Dirac equation to obtain  $\psi_1$  and hence  $\phi_1$ . Continuing in this way, we obtain sequences  $\{\phi_i\}$  and  $\{f_i\}$ . The appropriate estimates must be made if we are to show that this procedure converges. Below we will perform a calculation which suggests that uniform estimates for the warping factor  $\phi$  are possible. Once this is accomplished, it is a relatively easy task to uniformly bound the corresponding solution of the generalized Jang equation and the Dirac equation.

To begin, recall that the warping factor is set to be the norm squared of the Dirac spinor, that is,  $\phi = |\psi|^2$ . In the Appendix, it is shown that  $\phi$  solves an equation of the form

$$\bar{\Delta}\phi - \frac{1}{2}\bar{R}\phi = \mathcal{F}, \tag{26}$$

where  $\mathcal{F}$  is a function of first derivatives of the spinor  $\psi$  and first derivatives of  $\bar{g}$ . The scalar curvature has a nice form given by (19). In this formula

$$h_{ij} = \frac{\phi \nabla_{ij} f + \phi_i f_j + \phi_j f_i + \phi^2 \phi^l f_l f_j}{\sqrt{1 + \phi^2 |\nabla f|^2}}$$

is the second fundamental form of the Jang surface inside  $(M \times \mathbb{R}, g + \phi^2 dt^2)$ ,  $q$  and  $w$  are given by (17) and (18), and  $K$  is the extension to  $M \times \mathbb{R}$  of the initial data  $k$  given by

$$K(\partial_{x^i}, \partial_{x^j}) = k_{ij}, \quad K(\partial_{x^i}, \partial_t) = K(\partial_t, \partial_{x^i}) = 0, \quad K(\partial_t, \partial_t) = \frac{\langle \phi \nabla f, \phi \nabla f \rangle}{\sqrt{1 + \phi^2 |\nabla f|^2}}.$$

We observe that the scalar curvature  $\bar{R}$  contains two derivatives of  $\phi$  and three derivatives of  $f$ , and thus it appears that by applying the Schauder estimates to Eq. (26), the best we could hope for is an estimate of the form

$$|\phi|_{C^{2,\alpha}} \leq C(|\phi|_{C^{2,\alpha}} + |f|_{C^{3,\alpha}} + |\psi|_{C^{1,\alpha}} + |\bar{g}|_{C^{1,\alpha}}), \tag{27}$$

which of course is of no help at all. However, below, we shall calculate the divergence term in the expression for the scalar curvature, and invoke the generalized Jang equation, to show that this simple estimate may be improved.

Let  $\Gamma_{ij}^l$  and  $\bar{\Gamma}_{ij}^l$  be the Christoffel symbols for the initial data metric  $g$  and the Jang metric  $\bar{g}$ , respectively. Also note that

$$q_j = \frac{\phi f^l}{\sqrt{1 + \phi^2 |\nabla f|^2}} \left( \frac{\phi \nabla_{lj} f + \phi_l f_j + \phi_j f_l}{\sqrt{1 + \phi^2 |\nabla f|^2}} - k_{ij} \right).$$

Then a straightforward calculation yields

$$\begin{aligned} \bar{\text{div}}(\phi q) &= \bar{g}^{ij} \bar{\nabla}_i (\phi q_j) \\ &= \bar{g}^{ij} \phi_i q_j + \frac{\phi \bar{g}^{ij}}{1 + \phi^2 |\nabla f|^2} [2\phi \phi_i f^l \nabla_{lj} f + \phi^2 \nabla_{li} f \nabla_j^l f + \phi^2 f^l \nabla_i \nabla_l \nabla_j f \\ &\quad + \phi^2 f^l \nabla_{lm} f (\Gamma_{ij}^m - \bar{\Gamma}_{ij}^m) + \phi f^l \phi_i \bar{\nabla}_{ij} f + \phi f^l f_j \bar{\nabla}_{il} \phi + \phi \phi^l f_j \bar{\nabla}_{li} f \\ &\quad + \phi_i \phi_l f^l f_j + \phi_i \phi_j |\nabla f|^2 + \phi |\nabla f|^2 \bar{\nabla}_{ij} \phi + 2\phi f^l \phi_j \bar{\nabla}_{li} f \\ &\quad - \phi f^l (1 + \phi^2 |\nabla f|^2)^{-1} (\phi \nabla_{lj} f + \phi_l f_j + \phi_j f_l) (2\phi \phi_i |\nabla f|^2 + 2\phi^2 f^m \nabla_{mi} f)] \\ &\quad - \phi \bar{g}^{ij} \left( \frac{\phi f^l k_{lj}}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right). \end{aligned}$$

Observe that (see Ref. 11)

$$\Gamma_{ij}^m - \bar{\Gamma}_{ij}^m = \phi \phi^m f_i f_j - \frac{\phi f^m h_{ij}}{\sqrt{1 + \phi^2 |\nabla f|^2}},$$

and recall the Ricci commutation formula

$$\nabla_i \nabla_l \nabla_j f = \nabla_l \nabla_j \nabla_i f - R_{mjil} f^m.$$

Therefore, with the aid of the generalized Jang equation, we find

$$\begin{aligned} \frac{\phi \bar{g}^{ij}}{\sqrt{1 + \phi^2 |\nabla f|^2}} \nabla_l \nabla_j \nabla_i f &= \nabla_l \left( \frac{\phi \bar{g}^{ij} \nabla_{ij} f}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) - \nabla_{ij} f \nabla_l \left( \frac{\bar{g}^{ij} \phi}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) \\ &= \nabla_l \left[ \bar{g}^{ij} \left( k_{ij} - \frac{\phi_i f_j + \phi_j f_i}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) \right] - \nabla_{ij} f \nabla_l \left( \frac{\bar{g}^{ij} \phi}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) \\ &= -\nabla_{ij} f \nabla_l \left( \frac{\bar{g}^{ij} \phi}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) + \nabla_l (\bar{g}^{ij} k_{ij}) - \nabla_l \left( \frac{\bar{g}^{ij}}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right) (\phi_i f_j + \phi_j f_i) \\ &\quad - \bar{g}^{ij} \left( \frac{\nabla_{li} \phi f_j + \phi_i \nabla_{lj} f + \nabla_{jl} \phi f_i + \phi_j \nabla_{li} f}{\sqrt{1 + \phi^2 |\nabla f|^2}} \right). \end{aligned}$$

It follows that

$$\bar{\text{div}}(\phi q) = \frac{\phi^2}{1 + \phi^2 |\nabla f|^2} \left[ \left( g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2 |\nabla f|^2} \right) |\nabla f|^2 - \frac{f^i f^j}{1 + \phi^2 |\nabla f|^2} \right] \bar{\nabla}_{ij} \phi + \dots,$$

where  $\dots$  represents terms depending only on first derivatives of  $\phi$ , first derivatives of  $\bar{g}$ , and second derivatives of  $f$ .

This shows that  $\bar{\text{div}}(\phi q)$  is a degenerate elliptic operator for  $\phi$ , since the coefficients of the principal symbol may be rewritten as

$$g^{ij} |\nabla f|^2 - f^i f^j \geq 0.$$

Therefore, Eq. (26) is actually a strictly elliptic operator for  $\phi$ , with coefficients depending only on first derivatives of  $\phi$ , first derivatives of  $\bar{g}$ , and second derivatives of  $f$ . We may now apply the

Schauder estimates to obtain an improvement of the estimate (27):

$$|\phi|_{C^{2,\alpha}} \leq C(|\phi|_{C^{1,\alpha}} + |f|_{C^{2,\alpha}} + |\psi|_{C^{1,\alpha}} + |\bar{g}|_{C^{1,\alpha}}).$$

Moreover, it is clear from the structure of the generalized Jang equation and the Dirac equation that one should have

$$|f|_{C^{2,\alpha}} \leq C(1 + |\phi|_{C^{1,\alpha}}),$$

and

$$|\psi|_{C^{1,\alpha}} \leq C|\bar{g}|_{C^{1,\alpha}}.$$

Since  $\bar{g} = g + \phi^2 df^2$ , we also have

$$|\bar{g}|_{C^{1,\alpha}} \leq C(1 + |\phi|_{C^{1,\alpha}} + |f|_{C^{2,\alpha}}) \leq C(1 + |\phi|_{C^{1,\alpha}}).$$

Hence

$$|\phi|_{C^{2,\alpha}} \leq C(1 + |\phi|_{C^{1,\alpha}}).$$

These heuristic arguments yield strong evidence for a uniform bound on  $\phi$ , which as mentioned above, will lead to uniform bounds on  $f$  and  $\psi$ . Thus it appears that existence of a solution for the Dirac-Jang system is highly likely, and rests on a uniform  $C^{1,\alpha}$  bound for  $\phi$ .

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## APPENDIX: THE WARPING FACTOR

Here we derive Eq. (26) satisfied by the warping factor  $\phi = |\psi|^2$ . Write the connection (5) in the following way:

$$\nabla_i = \bar{\nabla}_i - A_i,$$

where  $\bar{\nabla}$  is the metric compatible (Levi-Civita) connection and

$$A_i = \frac{1}{2}c(\bar{E})c(e_i)c(e_0).$$

Direct computation yields

$$\begin{aligned} \bar{\Delta}|\psi|^2 &= \bar{\nabla}^i (\langle \bar{\nabla}_i \psi, \psi \rangle + \langle \psi, \bar{\nabla}_i \psi \rangle) \\ &= \bar{\nabla}^i (\langle (\nabla_i + A_i)\psi, \psi \rangle + \langle \psi, (\nabla_i + A_i)\psi \rangle) \\ &= \langle \bar{\nabla}^i (\nabla_i \psi), \psi \rangle + \langle (\bar{\nabla}^i A_i)\psi, \psi \rangle + \langle A_i \bar{\nabla}_i \psi, \psi \rangle \\ &\quad + \langle (\nabla_i + A_i)\psi, \bar{\nabla}_i \psi \rangle + \langle \bar{\nabla}^i \psi, (\nabla_i + A_i)\psi \rangle + \langle \psi, \bar{\nabla}^i (\nabla_i \psi) \rangle \\ &\quad + \langle \psi, A_i \bar{\nabla}^i \psi \rangle + \langle \psi, (\bar{\nabla}^i A_i)\psi \rangle \\ &= \langle \nabla^i \nabla_i \psi, \psi \rangle + \langle A^i \nabla_i \psi, \psi \rangle + \langle (\bar{\nabla}^i A_i)\psi, \psi \rangle \\ &\quad + \langle A_i \nabla^i \psi, \psi \rangle + \langle A_i A^i \psi, \psi \rangle + |\nabla \psi|^2 + \langle \nabla_i \psi, A^i \psi \rangle + \langle A_i \psi, \nabla^i \psi \rangle \\ &\quad + |A \psi|^2 + |\nabla \psi|^2 + \langle \nabla^i \psi, A_i \psi \rangle + \langle A^i \psi, \nabla_i \psi \rangle + |A \psi|^2 \\ &\quad + \langle \psi, A^i \nabla_i \psi \rangle + \langle \psi, A_i \nabla^i \psi \rangle + \langle \psi, A^i A_i \psi \rangle + \langle \psi, (\bar{\nabla}^i A_i)\psi \rangle. \end{aligned}$$

Now use the Lichnerowicz formula to obtain

$$\begin{aligned} \overline{\Delta}|\psi|^2 &= \frac{1}{8}(\langle \mathcal{R}\psi, \psi \rangle + \langle \psi, \mathcal{R}\psi \rangle) + 2|\nabla\psi|^2 + 2|A\psi|^2 \\ &\quad + 2(\langle A^i \nabla_i \psi, \psi \rangle + \langle \psi, A^i \nabla_i \psi \rangle) \\ &\quad + 2(\langle \nabla_i \psi, A^i \psi \rangle + \langle A_i \psi, \nabla^i \psi \rangle) \\ &\quad + \langle (\overline{\nabla}^i A_i)\psi, \psi \rangle + \langle \psi, (\overline{\nabla}^i A_i)\psi \rangle \\ &\quad + \langle A_i A^i \psi, \psi \rangle + \langle \psi, A_i A^i \psi \rangle, \end{aligned}$$

where

$$\mathcal{R} = \overline{R} - 2|\overline{E}|^2 - (\operatorname{div} \overline{E})c(e_0).$$

In what follows, for convenience we will neglect the notation  $c(e_i)\psi$  for Clifford multiplication, and instead simply write  $e_i\psi$ . Observe that

$$\begin{aligned} \langle \overline{E}e_i e_0 \overline{E}e_i e_0 \psi, \psi \rangle &= -\langle e_i e_0 \overline{E}e_i e_0 \psi, \overline{E}\psi \rangle \\ &= \langle e_0 \overline{E}e_i e_0 \psi, e_i \overline{E}\psi \rangle \\ &= \langle \overline{E}e_i e_0 \psi, e_0 e_i \overline{E}\psi \rangle \\ &= -\langle \overline{E}e_i e_0 \psi, e_i e_0 \overline{E}\psi \rangle \\ &= \langle \overline{E}e_i e_0 \psi, e_i \overline{E}e_0 \psi \rangle \\ &= \langle \overline{E}e_i e_0 \psi, (-\overline{E}e_i - 2\overline{g}(\overline{E}, e_i))e_0 \psi \rangle \\ &= -\langle \overline{E}e_i e_0, \overline{E}e_i e_0 \psi \rangle - 2\overline{g}(\overline{E}, e_i)\langle \overline{E}e_i e_0 \psi, e_0 \psi \rangle. \end{aligned}$$

Here we have used the basic law of Clifford multiplication, namely

$$c(X)c(Y) + c(Y)c(X) = -2\tilde{g}(X, Y),$$

where  $\tilde{g}$  denotes the spacetime metric. Furthermore, we also observe that

$$\begin{aligned} \langle \overline{E}e_i e_0 \psi, e_0 \psi \rangle &= \langle e_0 \overline{E}e_i e_0 \psi, \psi \rangle \\ &= -\langle \overline{E}e_0 e_i e_0 \psi, \psi \rangle \\ &= \langle \overline{E}e_i e_0 e_0 \psi, \psi \rangle \\ &= \langle \overline{E}e_i \psi, \psi \rangle, \end{aligned}$$

and therefore

$$\overline{g}(\overline{E}, e_i)\langle \overline{E}e_i e_0 \psi, e_0 \psi \rangle = \overline{g}(\overline{E}, e_i)\langle \overline{E}e_i \psi, \psi \rangle = \langle \overline{E}\overline{E}\psi, \psi \rangle = |\overline{E}|^2|\psi|^2.$$

It follows that

$$\langle A_i A^i \psi, \psi \rangle = \langle \psi, A_i A^i \psi \rangle = -|A\psi|^2 + \frac{1}{2}|\overline{E}|^2|\psi|^2.$$

Moreover

$$\overline{\nabla}_i A_j = \frac{1}{2}(\overline{\nabla}_i \overline{E})e_j e_0 + \frac{1}{2}\overline{E}(\overline{\nabla}_i e_j)e_0,$$

since  $\overline{\nabla}_i e_0 = 0$ , as the Jang initial data are assumed to have no extrinsic curvature. Also

$$\begin{aligned} \langle (\overline{\nabla}_i \overline{E})e_j e_0 \psi, \psi \rangle &= -\langle \psi, (\overline{\nabla}_i \overline{E})e_j e_0 \psi \rangle - 2\overline{g}(\overline{\nabla}_i \overline{E}, e_j)\langle \psi, e_0 \psi \rangle \\ &= -\langle \psi, (\overline{\nabla}_i \overline{E})e_j e_0 \psi \rangle - 2(\overline{\nabla}_i \overline{E}_j - \overline{g}(\overline{E}, \overline{\nabla}_i e_j))\langle \psi, e_0 \psi \rangle, \end{aligned}$$

and

$$\langle \overline{E}(\overline{\nabla}_i e_j)e_0 \psi, \psi \rangle = -\langle \psi, \overline{E}(\overline{\nabla}_i e_j)e_0 \psi \rangle - 2\overline{g}(\overline{E}, \overline{\nabla}_i e_j)\langle \psi, e_0 \psi \rangle.$$

Hence

$$\langle (\bar{\nabla}^i A_i) \psi, \psi \rangle = -\langle \psi, (\bar{\nabla}^i A_i) \psi \rangle - (\bar{\text{div}} \bar{E}) \langle \psi, e_0 \psi \rangle.$$

Finally, we obtain

$$\begin{aligned} \bar{\Delta} |\psi|^2 &= \frac{1}{8} (\langle \mathcal{R} \psi, \psi \rangle + \langle \psi, \mathcal{R} \psi \rangle) \\ &\quad + |\bar{E}|^2 |\psi|^2 + 2 |\nabla \psi|^2 - (\bar{\text{div}} \bar{E}) \langle \psi, e_0 \psi \rangle \\ &\quad + 2 (\langle A^i \nabla_i \psi, \psi \rangle + \langle \psi, A^i \nabla_i \psi \rangle) \\ &\quad + 2 (\langle \nabla_i \psi, A^i \psi \rangle + \langle A_i \psi, \nabla^i \psi \rangle). \end{aligned}$$

It follows that  $\phi$  satisfies Eq. (26) with a right-hand side given by

$$\begin{aligned} \mathcal{F} &= \frac{1}{8} (\langle \mathcal{R} \psi, \psi \rangle + \langle \psi, \mathcal{R} \psi \rangle) - \frac{1}{2} \bar{R} |\psi|^2 \\ &\quad + |\bar{E}|^2 |\psi|^2 + 2 |\nabla \psi|^2 - (\bar{\text{div}} \bar{E}) \langle \psi, e_0 \psi \rangle \\ &\quad + 2 (\langle A^i \nabla_i \psi, \psi \rangle + \langle \psi, A^i \nabla_i \psi \rangle) \\ &\quad + 2 (\langle \nabla_i \psi, A^i \psi \rangle + \langle A_i \psi, \nabla^i \psi \rangle). \end{aligned}$$

Notice that in the case of equality for the positive mass theorem with charge,  $\mathcal{F} = 0$  so that (26) reduces to the correct equation for the warping factor  $\phi$ . Namely, in this case the spinor  $\psi$  is covariantly constant with respect to the connection  $\nabla$ ,  $\mathcal{R} = 0$ , and (23) holds. Thus we obtain

$$\bar{\Delta} \phi - |\bar{E}|^2 \phi = 0.$$

This is the correct equation for the warping factor of a static spacetime satisfying the Einstein-Maxwell equations. To see this, observe that if  $\tilde{g} = -\phi^2 dt^2 + \bar{g}$  is the static spacetime metric, then its scalar curvature is given by

$$\tilde{R} = \bar{R} - 2\phi^{-1} \bar{\Delta} \phi.$$

If  $\tilde{g}$  satisfies the Einstein-Maxwell equations then

$$\tilde{R} = -\tilde{g}^{ab} T_{ab},$$

where  $T$  is the stress-energy tensor. Since

$$T^{ab} = - \left( F^{ac} F_c^b + \frac{1}{4} \tilde{g}^{ab} F_{cd} F^{cd} \right)$$

where  $F_{ab}$  is the field strength tensor of the electro-magnetic field, we have  $\tilde{R} = 0$  so that

$$\bar{\Delta} \phi - \frac{1}{2} \bar{R} \phi = 0.$$

But for (time symmetric) electro-vacuum initial data, we have  $\bar{R} = 2|\bar{E}|^2$ , which confirms that our choice of warping factor satisfies the correct equation. This is quite amazing, since the choice  $\phi = |\psi|^2$  was based on an entirely different motivation. Namely,  $\phi$  was chosen in order to allow the divergence term in  $\bar{R}$  to be integrated away in the Lichnerowicz formula.

<sup>1</sup>P. F. Yip, "A strictly-positive mass theorem," *Commun. Math. Phys.* **108**, 653–665 (1987).

<sup>2</sup>R. Beig and P. T. Chrusciel, "Killing vectors in asymptotically flat space-times: 1. Asymptotically translational Killing vectors and the rigid positive energy theorem," *J. Math. Phys.* **37**, 1939–1961 (1996); e-print [arXiv:gr-qc/9510015](https://arxiv.org/abs/gr-qc/9510015).

<sup>3</sup>R. Schoen and S. T. Yau, "On the proof of the positive mass conjecture in general relativity," *Commun. Math. Phys.* **65**, 45–76 (1979).

<sup>4</sup>R. Schoen and S. T. Yau, "Proof of the positive mass theorem. II," *Commun. Math. Phys.* **79**, 231–260 (1981).

<sup>5</sup>E. Witten, "A new proof of the positive energy theorem," *Commun. Math. Phys.* **80**, 381–402 (1981).

<sup>6</sup>J. M. Lee and T. H. Parker, "The Yamabe problem," *Bull. Am. Math. Soc.* **17**, 37–91 (1987).

- <sup>7</sup>G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, "Positive mass theorems for black holes," *Commun. Math. Phys.* **88**, 295–308 (1983).
- <sup>8</sup>R. A. Bartnik and P. T. Chruściel, "Boundary value problems for Dirac-type equations," *J. Reine Angew. Math.* **579**, 13–73 (2005).
- <sup>9</sup>P. T. Chruściel, H. S. Reall, and P. Tod, "On Israel-Wilson-Perjés black holes," *Class. Quantum Grav.* **23**, 2519–2540 (2006).
- <sup>10</sup>H. L. Bray and M. A. Khuri, "A Jang equation approach to the Penrose inequality," *Discrete Contin. Dyn. Syst.* **27**, 741–766 (2010).
- <sup>11</sup>H. L. Bray and M. A. Khuri, "P.D.E.'s which imply the Penrose conjecture," *Asian J. Math.* **15**, 557–610 (2011).
- <sup>12</sup>Q. Han and M. Khuri, "Existence and blow-up behavior for solutions of the generalized Jang equation," *Comm. Partial Differential Equations* (to appear); e-print [arXiv:1206:0079](https://arxiv.org/abs/1206.0079) [math.DG].
- <sup>13</sup>T. Parker and C. H. Taubes, "On Witten's proof of the positive energy theorem," *Commun. Math. Phys.* **84**, 223–238 (1982).
- <sup>14</sup>M. M. Disconzi and M. A. Khuri, "On the Penrose inequality for charged black holes," *Class. Quantum Grav.* **29**, 245019 (2012).