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# A Penrose-Like Inequality for General Initial Data Sets

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**Abstract:** We establish a Penrose-Like Inequality for general (not necessarily time symmetric) initial data sets of the Einstein equations which satisfy the dominant energy condition. More precisely, it is shown that the ADM energy is bounded below by an expression which is proportional to the square root of the area of the outermost future (or past) apparent horizon.

## 1. Introduction

In an attempt to find a counterexample for his Cosmic Censorship Conjecture, R. Penrose [12] proposed a necessary condition for its validity, in the form of an inequality relating the ADM mass and area of any event horizon in an asymptotically flat spacetime:

$$\text{Mass} \geq \sqrt{\frac{\text{Area}}{16\pi}}. \tag{1.1}$$

Unfortunately this Penrose Inequality can only be proven with knowledge of the full spacetime development, as otherwise it would not be possible to locate the event horizon in a given spacelike slice. Thus it is customary to reformulate (1.1) so that the quantities involved may be calculated solely from local information, namely initial data sets for the Einstein equations. By an initial data set we are referring to a triple $(M, g, k)$, consisting of a Riemannian 3-manifold $M$ with metric $g$ and a symmetric 2-tensor $k$ representing the extrinsic curvature of a spacelike slice. These data are required to satisfy the constraint equations

$$\begin{aligned} 16\pi\mu &= R + (\mathrm{Tr}_g k)^2 - |k|^2, \\ 8\pi J_i &= \nabla^j (k_{ij} - (\mathrm{Tr}_g k) g_{ij}), \end{aligned}$$

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where $R$ is scalar curvature and $\mu$, $J$ are respectively the energy and momentum densities for the matter fields. If all measured energy densities are nonnegative then $\mu \geq |J|$, which will be referred to as the dominant energy condition. Moreover the initial data set will be taken to be asymptotically flat (with one end), so that at spatial infinity the metric and extrinsic curvature satisfy the following fall-off conditions:

$$|\partial^l(g_{ij}-\delta_{ij})| = O(r^{-l-1}), \quad |\partial^l k_{ij}| = O(r^{-l-2}), \quad l = 0, 1, 2, \quad \text{as} \quad r \to \infty.$$

The ADM energy and momentum are then well defined by

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\nu^j, \quad P_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k_{ij} - (\mathrm{Tr}_g k) g_{ij})\nu^j,$$

where $S_r$ are coordinate spheres in the asymptotic end with unit outward normal $\nu$.

The strength of the gravitational field in the vicinity of a 2-surface $\Sigma \subset M$ may be measured by the null expansions

$$\theta_\pm := H_\Sigma \pm \mathrm{Tr}_\Sigma k,$$

where $H_\Sigma$ is the mean curvature with respect to the unit outward normal (pointing towards spatial infinity). The null expansions measure the rate of change of area for a shell of light emitted by the surface in the outward future direction ($\theta_+$), and outward past direction ($\theta_-$). Thus the gravitational field is interpreted as being strong near $\Sigma$ if $\theta_+ < 0$ or $\theta_- < 0$, in which case $\Sigma$ is referred to as a future (past) trapped surface. Future (past) apparent horizons arise as boundaries of future (past) trapped regions and satisfy the equation $\theta_+ = 0$ ($\theta_- = 0$). In the setting of the initial data set formulation of the Penrose Inequality, apparent horizons take the place of event horizons, in that the area of the event horizon is replaced by the area of the outermost apparent horizon (or in some formulations by the least area required to enclose an apparent horizon).

The Penrose Inequality has been established by Huisken & Ilmanen [8] and by Bray [2] in the time symmetric case, that is when $k = 0$. At the present time the conjecture for arbitrary initial data sets remains open, however recently Bray and the author [3] have succeeded in reducing this problem to the question of existence for a canonical system of partial differential equations. The purpose here is to establish a Penrose-Like Inequality for arbitrary initial data satisfying the dominant energy condition. This new inequality will generalize the following one obtained by Herzlich [7] in the time symmetric case.

**Theorem 1.1.** *Let $(M, g)$ be a 3-dimensional asymptotically flat Riemannian manifold with nonnegative scalar curvature, and boundary consisting of a minimal 2-sphere with area $|\partial M|$. Then the ADM energy satisfies*

$$\mathrm{E}(g) \geq \frac{\sigma}{2(1+\sigma)}\sqrt{\frac{|\partial M|}{\pi}},$$

*where*

$$\sigma = \sqrt{\frac{|\partial M|}{\pi}} \inf_{\substack{v\in C_c^\infty \\ v\neq 0}} \frac{\| \nabla v \|^2_{L^2(M)}}{\| v \|^2_{L^2(\partial M)}}.$$

*Furthermore equality holds if and only if $(M, g)$ is a portion of the $t = 0$ slice of the Schwarzschild spacetime with mass $\sqrt{|\partial M|/16\pi}$.*

A useful device for extending results in the time symmetric case to the general case is Jang's deformation [9] of the initial data, which was successfully employed by Schoen and Yau [13] in their proof of the Positive Energy Theorem. In their application, special solutions of Jang's equation which exhibit blow-up behavior at apparent horizons played an integral role, and for some time it has been suggested that these solutions may be helpful in studying the Penrose Inequality (see [10] for some problems that can occur with this approach). For this to be the case, solutions which blow-up at a given apparent horizon must always be shown to exist. In fact such a theorem has recently been established by Metzger in [11]. More precisely Metzger has shown that given an initial data set containing an outermost future (or past) apparent horizon, there exists a smooth solution of Jang's equation outside of the outermost apparent horizon which blows-up to $+\infty$ ($-\infty$) in the form of a cylinder over the horizon, and vanishes at spatial infinity. Here an outermost future (past) apparent horizon refers to a future (past) apparent horizon outside of which there is no other apparent horizon; such a horizon may have several components. We will denote the Jang surface associated with the given blow-up solution of Jang's equation by $(\overline{M}, \overline{g})$, and its connection by $\overline{\nabla}$. We will show

**Theorem 1.2.** *Let $(M, g, k)$ be an asymptotically flat initial data set for the Einstein equations satisfying the dominant energy condition $\mu \geq |J|$. If the boundary consists of an outermost future (past) apparent horizon with components of area $|\partial_i M|$, $i = 1, \ldots, n$, then the ADM energy satisfies*

$$\mathrm{E}(g) \geq \frac{\sigma}{2(1+\sigma)} \sum_{i=1}^{n} \sqrt{\frac{|\partial_i M|}{\pi}},$$

*where*

$$\sigma = \left( \sum_{i=1}^{n} \sqrt{4\pi |\partial_i M|} \right)^{-1} \inf \| \overline{\nabla} v \|^2_{L^2(\overline{M})},$$

*with the infimum taken over all $v \in C^\infty(M)$ such that $v(x) \to 0$ as $x \to \partial M$ and $v(x) \to 1$ as $|x| \to \infty$.*

*Remark 1.3.* Although the hypotheses require a boundary consisting entirely of future or entirely of past apparent horizons, our proof gives a bit more. Namely when both types are present the same result holds, where $\{\partial_i M\}_{i=1}^n$ consists entirely of future or entirely of past apparent horizons.

An important point to note concerning Theorem 1.2 is that the case of equality is not considered. The reasons for this are the following. First the Jang equation is designed to embed the initial data into Minkowski space if equality were to occur (as is done in the Positive Energy Theorem), and so there is no chance of obtaining and embedding into the Schwarzchild spacetime in this situation, as the Penrose Inequality demands. Moreover, it will in fact be shown that the case of equality can never be achieved. This implies that the current result is not optimal (unlike Theorem 1.1), and suggests that there may be a better choice of boundary conditions for the Jang equation which does yield an optimal result.

Another point to note is that the constant $\sigma$ is dimensionless, and so is actually independent of the area of the boundary $\partial M$. Furthermore $\sigma$ never vanishes, and therefore Theorem 1.2 does give a positive lower bound for the ADM mass in terms of the area of the apparent horizon, which is consistent with the spirit of the Penrose Inequality. Moreover the theorem may be generalized to the setting of initial data containing a trapped surface, to give a positive lower bound for the ADM mass in terms of the least area required to enclose the trapped surface. To see this recall that Andersson and Metzger [1], and Eichmair [4], have shown that the existence of a future (past) trapped surface in an asymptotically flat initial data set implies the existence of an outermost future (past) apparent horizon. One may then apply Theorem 1.2 to obtain the desired result.

The proof of Theorem 1.2 closely follows that of Theorem 1.1. The main difference, or new idea, is to employ blow-up solutions for Jang's equation in an appropriate way. However the argument still relies on the following version of the Positive Energy Theorem due to Herzlich.

**Theorem 1.4** [7]. *Let $(M, g)$ be a 3-dimensional asymptotically flat Riemannian manifold with nonnegative scalar curvature. If the boundary $\partial M$ consists of n components having spherical topology and mean curvature (calculated with respect to the normal pointing inside $M$) satisfying $H_{\partial_i M} \leq \sqrt{16\pi/|\partial_i M|}$, $1 \leq i \leq n$, then $E(g) \geq 0$ and when equality occurs $g$ is flat.*

*Remark 1.5.* The statement of Herzlich's original theorem only allowed the boundary $\partial M$ to have one component. However the same spinor proof may easily be extended to allow for finitely many components as in Theorem 1.4.

In continuing with the outline of proof for Theorem 1.2, there are three primary steps. The first is to deform the initial data by constructing a blow-up solution of the Jang equation, which as mentioned above has already been established. This deformation yields a positivity property for the scalar curvature of the Jang metric $\overline{g}$. The next step entails cutting off the cylindrical ends of the blown-up Jang surface at a height $T$ to obtain a manifold with boundary $\overline{M}_T$, and then making a conformal deformation $(\overline{M}_T, \widehat{g}_T := u_T^4 \overline{g})$ to obtain a manifold with zero scalar curvature and with each boundary component satisfying $\widehat{H}_{\partial_i \overline{M}_T} = \sqrt{16\pi/|\partial_i \overline{M}_T|_{\widehat{g}_T}}$, $1 \leq i \leq n$. Existence of a conformal factor $u_T$ satisfying these properties will be established by a variational argument, which heavily depends on the positivity property for the scalar curvature of the Jang metric as well as the blow-up behavior of the Jang surface at the horizon. One may then undertake the last step, which consists of applying Theorem 1.4 to obtain $E(\widehat{g}_T) \geq 0$. The desired lower bound for $E(g) = E(\overline{g})$ is then produced by estimating the difference $E(\overline{g}) - E(\widehat{g}_T)$ and letting $T \to \infty$.

## 2. The Jang Surface

The goal of this section is to give a precise description of the blow-up solution to the Jang equation, as well as to record certain qualitative properties of the resulting Jang surface. Let us first recall some basic facts. The Jang surface $\overline{M}$ is given by a graph $t = f(x)$ in the product manifold $(M \times \mathbb{R}, g + dt^2)$, and so has induced metric $\overline{g} = g + df^2$. The function $f$ is required to satisfy the Jang equation:

$$\overline{g}^{ij}\left(\frac{\nabla_{ij} f}{\sqrt{1 + |\nabla_g f|^2}} - k_{ij}\right) = 0. \tag{2.1}$$

Here $\nabla_{ij}$ denote second covariant derivatives with respect to $g$ and

$$\overline{g}^{ij} = g^{ij} - \frac{f^i f^j}{1 + |\nabla_g f|^2}$$

is the inverse matrix for $\overline{g}_{ij}$ with $f^i = g^{ij}\nabla_i f$, and therefore Jang's equation simply asserts that the mean curvature of the graph is equal to the trace of $k$ over the graph (assuming that the tensor $k$ has been extended trivially to all of $M \times \mathbb{R}$). The motivation for solving Jang's equation is to obtain a positivity property for the scalar curvature of the Jang surface. In particular, if $f$ satisfies Eq. (2.1) then the scalar curvature of $\overline{g}$ has the following expression (see [13]):

$$\overline{R} = 16\pi(\mu - J(w)) + |\overline{h} - k|^2_{\overline{g}} + 2|q|^2_{\overline{g}} - 2\mathrm{div}_{\overline{g}}(q), \tag{2.2}$$

where

$$w_i = \frac{\nabla_i f}{\sqrt{1 + |\nabla_g f|^2}}, \quad q_i = \frac{f^j}{\sqrt{1 + |\nabla_g f|^2}}(\overline{h}_{ij} - k_{ij}),$$

and $\overline{h}$ is the second fundamental form of $\overline{M}$. In addition to the positivity property for the scalar curvature, we will require the Jang surface to exhibit blow-up behavior at $\partial M$ in order to construct the conformal factor described in the introduction. It turns out that such a solution always exists as long as $\partial M$ is an outermost horizon.

**Theorem 2.1** [11]. *Suppose that $\partial M$ is an outermost future (past) apparent horizon. Then there exists an open set $\Omega \subset M$ (with $(M - \Omega) \cap \partial M = \emptyset$) and a smooth function $f : \Omega \to \mathbb{R}$ satisfying (2.1), such that $\partial\Omega - \partial M$ consists of past (future) apparent horizons, $\overline{M} = \mathrm{graph}(f)$ is asymptotic to the cylinders $\partial M \times \mathbb{R}_+$ ($\partial M \times \mathbb{R}_-$) and $\partial\Omega \times \mathbb{R}_-$ ($\partial\Omega \times \mathbb{R}_+$), and $f(x) \to 0$ as $|x| \to \infty$.*

This theorem yields the desired blow-up behavior at $\partial M$ with the added feature that blow-up may occur elsewhere at $\partial\Omega - \partial M$ as well, if $M$ contains apparent horizons of the other type (with respect to $\partial M$). However the hypotheses of Theorem 1.2 do not allow for such extra horizons, so that in fact $\Omega = M$. We remark that the sole reason for prohibiting extra horizons in the initial data is to ensure that each component of $\partial\Omega$ has spherical topology, which is needed when applying the Positive Energy Theorem, Theorem 1.4. Thus one could allow apparent horizons of the other type, if they have spherical topology.

The other goal of this section is to record the decay rate for certain geometric quantities associated with the Jang surface. Since the solution of Jang's equation blows-up at $\partial M$ in the form of a cylinder, in a neighborhood of each boundary component the Jang surface may be foliated by the level sets $t = f(x)$, which we denote by $\overline{\Sigma}_t$. Similarly in a neighborhood of each boundary component, $M$ may be foliated by the projection of the level sets $\overline{\Sigma}_t$ onto $M$, which we denote by $\Sigma_t$. We can then introduce coordinates $(r, \xi^2, \xi^3)$ in such a neighborhood of each component, where $r = |t|^{-1}$ and $\xi^2, \xi^3$ are coordinates on a 2-sphere. Note that as $r \to 0$ the projections $\Sigma_r$ converge to their associated component of $\partial M$. Furthermore the $r$-coordinate may be chosen orthogonal to its level sets, so that the initial data metric takes the form

$$g = g_{11}dr^2 + \sum_{i,j=2}^{3} g_{ij}d\xi^i d\xi^j.$$

**Lemma 2.2.** *Consider the level sets $\overline{\Sigma}_r$ of the blown-up Jang surface near a component of $\partial M$. If $\overline{H}_{\overline{\Sigma}_r}$ denotes the mean curvature of $\overline{\Sigma}_r$ with respect to the inward pointing (towards spatial infinity) normal $\overline{N}$, then $\overline{H}_{\overline{\Sigma}_r} - q(\overline{N}) \to 0$ as $r \to 0$.*

*Proof.* A calculation in [14] (p. 10) shows that

$$\overline{H}_{\overline{\Sigma}_r} - q(\overline{N}) = \sqrt{1+|\nabla_g f|^2}(H_{\Sigma_r} \pm \mathrm{Tr}_{\Sigma_r} k) \mp \frac{\mathrm{Tr}_{\Sigma_r} k}{|\nabla_g f| + \sqrt{1+|\nabla_g f|^2}},$$

where $H_{\Sigma_r}$ is the mean curvature of $\Sigma_r$, $\mathrm{Tr}_{\Sigma_r} k$ is the trace over $\Sigma_r$, and + (−) is chosen depending on whether the particular component of $\partial M$ in question is a future (past) horizon respectively. From this expression we see that it is enough to show that the first term on the right-hand side approaches zero as $r \to 0$. Fortunately this same expression appears in the Jang equation, and yields the desired result. To see this, write the Jang equation in the coordinates $(r, \xi^2, \xi^3)$ to obtain

$$\begin{aligned} &\frac{g^{11}}{1+g^{11}f_{,r}^2}(f_{,rr} - \Gamma_{11}^1 f_{,r}) - \sum_{i,j=2}^{3} g^{ij}\Gamma_{ij}^1 f_{,r} \\ &\quad = \sqrt{1+g^{11}f_{,r}^2}\left(\frac{g^{11}}{1+g^{11}f_{,r}^2}k_{11} + \sum_{i,j=2}^{3} g^{ij}k_{ij}\right), \end{aligned}$$

where $f_{,r}$, $f_{,rr}$ are partial derivatives and $\Gamma_{ij}^1$ are Christoffel symbols for $g$ given by

$$\Gamma_{11}^1 = \frac{1}{2}g^{11}\partial_r g_{11}, \quad \Gamma_{ij}^1 = -\sqrt{g^{11}}h_{ij}, \quad 2 \le i, j \le 3,$$

with $h_{ij}$ denoting the second fundamental form of $\Sigma_r$. It follows that

$$\sqrt{1+|\nabla_g f|^2}(H_{\Sigma_r} \pm \mathrm{Tr}_{\Sigma_r} k) = \pm\frac{g^{11}f_{,rr}}{1+g^{11}f_{,r}^2} + O\left(\frac{1}{\sqrt{1+|\nabla_g f|^2}}\right).$$

Lastly we observe that by definition of the coordinate $r$, $f(r) = \pm r^{-1}$, and therefore

$$\sqrt{1+|\nabla_g f|^2}(H_{\Sigma_r} \pm \mathrm{Tr}_{\Sigma_r} k) = O(r) \quad \text{as} \quad r \to 0.$$

□

**Lemma 2.3.** *The solution of Jang's equation satisfies the following fall-off condition at spatial infinity:*

$$|\nabla^l f|(x) = O(|x|^{-\frac{1}{2}-l}) \quad as \quad |x| \to \infty, \quad l = 0, 1, 2.$$

*In particular, the energy of the Jang metric $\overline{g}$ equals the energy of $g$.*

*Proof.* See Schoen and Yau [13]. □

## 3. The Conformal Factor

In this section we will complete the last preliminary step before application of the Positive Energy Theorem. Namely we will conformally deform the Jang metric to zero scalar curvature on a portion of the Jang surface, while at the same time prescribing the mean curvature of its boundary. The region to be considered consists of the portion of the Jang surface lying between the horizontal planes $t = \pm T$, and will be denoted by $\overline{M}_T$. We then search for a conformal factor $u_T$ satisfying the following boundary value problem:

$$\overline{\Delta} u_T - \tfrac{1}{8}\overline{R} u_T = 0 \quad \text{on} \quad \overline{M}_T, \tag{3.1}$$
$$\partial_{\overline{\nu}} u_T + \tfrac{1}{4}\overline{H}_{\partial_i \overline{M}_T} u_T = \tfrac{1}{4}\sqrt{\tfrac{16\pi}{|\partial_i \overline{M}_T|_{\widehat{g}_T}}}\, u_T^3 \quad \text{on each } i\text{th component of} \quad \partial\overline{M}_T,$$
$$u_T = 1 + \tfrac{A_T}{|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty,$$

where $A_T$ is a constant and $\overline{H}_{\partial_i \overline{M}_T}$ denotes mean curvature with respect to the unit inward normal $\overline{\nu}$ (pointing inside $\overline{M}_T$). This ensures that $(\overline{M}_T, \widehat{g}_T := u_T^4 \overline{g})$ has zero scalar curvature $\widehat{R} \equiv 0$ and mean curvature on each $i^{\text{th}}$ component of $\partial\overline{M}_T$ given by $\widehat{H}_{\partial_i \overline{M}_T} = \sqrt{16\pi/|\partial_i \overline{M}_T|_{\widehat{g}_T}}$. These two properties, combined with the fact that each component of $\partial M$ must have spherical topology ([5,6]), then guarantee that Theorem 1.4 is applicable.

We shall use a variational argument, just as in [7], to construct $u_T := 1 + v_T$. In this regard observe that boundary value problem (3.1) arises as the Euler-Lagrange equation for the functional

$$Q(v) = \frac{1}{2}\int_{\overline{M}_T}\left(|\overline{\nabla} v|^2 + \frac{1}{8}\overline{R}(1+v)^2\right) + \frac{\sqrt{\pi}}{2}\left(\int_{\partial\overline{M}_T}(1+v)^4\right)^{1/2}$$
$$-\frac{1}{8}\int_{\partial\overline{M}_T}\overline{H}_{\partial\overline{M}_T}(1+v)^2.$$

More precisely, we will search for a global minimum over the weighted Sobolev space

$$W^{1,2}_{-1}(\overline{M}_T) = \{v \in W^{1,2}_{loc}(\overline{M}_T) \mid |x|^{l-1}\overline{\nabla}^l v \in L^2(\overline{M}_T),\ \ l = 0, 1\}.$$

**Theorem 3.1.** *Given $T > 0$ sufficiently large, there exists a function $v_T \in W^{1,2}_{-1}(\overline{M}_T) \cap C^\infty(\overline{M}_T)$ at which $Q$ attains a global minimum. Moreover $u_T = 1 + v_T$ never vanishes and satisfies the asymptotic behavior in* (3.1).

*Proof.* In order to establish the existence (as well as the regularity and asymptotic behavior) portion of this theorem it is enough, by the arguments of [7], to show that for $T$ sufficiently large the functional $Q$ is nonnegative. To see this use formula (2.2) and integrate the divergence term by parts to find that for any $v \in W^{1,2}_{-1}(\overline{M}_T)$,

$$Q(v) \geq \int_{\overline{M}_T}\left(\frac{3}{8}|\overline{\nabla} v|^2 + \pi(\mu - |J|)(1+v)^2\right) + \frac{\sqrt{\pi}}{2}\left(\int_{\partial\overline{M}_T}(1+v)^4\right)^{1/2}$$
$$-\frac{1}{8}\int_{\partial\overline{M}_T}(\overline{H}_{\partial\overline{M}_T} - q(\overline{N}))(1+v)^2. \tag{3.2}$$

By Lemma 2.2 $\overline{H}_{\partial\overline{M}_T} - q(\overline{N}) = O(T^{-1})$, and a calculation shows that the area of $\partial\overline{M}_T$ agrees with the area of $\Sigma_T \subset M$ which remains bounded as $T \to \infty$. It then follows from Jensen's Inequality,

$$\left(\int_{\partial\overline{M}_T} (1+v)^2\right)^2 \leq |\partial\overline{M}_T| \int_{\partial\overline{M}_T} (1+v)^4,$$

that for $T$ sufficiently large $Q$ is nonnegative.

It remains to show that $u_T = 1 + v_T$ is strictly positive. So suppose that $u_T$ is not positive and let $D_-$ be the domain on which $u_T < 0$. Since $u_T \to 1$ as $|x| \to \infty$, the closure of $D_-$ must be compact. Now multiply Eq. (3.1) through by $u_T$ and integrate by parts to obtain

$$\int_{D_-} |\overline{\nabla} u_T|^2 \leq 0.$$

Note that if $D_- \cap \partial\overline{M}_T \neq \emptyset$, then the same arguments used above to show that $Q$ is nonnegative, must be employed. It follows that $u_T \geq 0$. To show that $u_T > 0$, one need only apply Hopf's Maximum Principle (the boundary condition of (3.1) must be used to obtain this conclusion at $\partial\overline{M}_T$). $\square$

## 4. Proof of Theorem 1.2

Here we shall carry out the last step in the proof of Theorem 1.2, namely to apply the Positive Energy Theorem and to compare the two energies $\mathrm{E}(g)$ and $\mathrm{E}(\widehat{g}_T)$. Observe that all the hypotheses of Theorem 1.4 are satisfied by $(\overline{M}_T, \widehat{g}_T)$ so that $\mathrm{E}(\widehat{g}_T) \geq 0$. Therefore a straightforward calculation yields

$$\mathrm{E}(g) \geq \mathrm{E}(g) - \mathrm{E}(\widehat{g}_T) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{|x|=r} \partial_{\overline{\nu}} u_T. \tag{4.1}$$

Furthermore upon integrating by parts and using boundary value problem (3.1) we obtain

$$\lim_{r\to\infty} \int_{|x|=r} \partial_{\overline{\nu}} u_T = \lim_{r\to\infty} \int_{|x|=r} u_T \partial_{\overline{\nu}} u_T = 2Q(v_T). \tag{4.2}$$

Now suppose that $Q(v_T) \leq \eta \sum_{i=1}^{n} \sqrt{\pi |\partial_i \overline{M}_T|}$ for some positive constant $\eta$, where $n$ denotes the number of components comprising $\partial M$. Then integrating by parts, and using arguments such as those found in the proof of Theorem 3.1, shows that there exists a constant $C > 0$ independent of $T$ such that

$$\frac{3}{8}\int_{\overline{M}_T} |\overline{\nabla} v_T|^2 + \left(\frac{1 - CT^{-1}}{2}\right) \sum_{i=1}^{n} \sqrt{\frac{\pi}{|\partial_i \overline{M}_T|}} \int_{\partial_i \overline{M}_T} (1+v_T)^2 \leq \eta \sum_{i=1}^{n} \sqrt{\pi |\partial_i \overline{M}_T|}.$$

However by Young's Inequality,

$$(1+v_T)^2 \geq 1 - \frac{1}{\delta} + (1-\delta) v_T^2$$

for any $\delta > 0$, and therefore

$$\frac{3}{8}\int_{\overline{M}_T}|\overline{\nabla}v_T|^2 + (1-\delta)\left(\frac{1-CT^{-1}}{2}\right)\sum_{i=1}^{n}\sqrt{\frac{\pi}{|\partial_i\overline{M}_T|}}\int_{\partial_i\overline{M}_T}v_T^2$$
$$\leq (\eta - \frac{1}{2}(1-\delta^{-1})(1-CT^{-1}))\sum_{i=1}^{n}\sqrt{\pi|\partial_i\overline{M}_T|}.$$

It follows that the left-hand side is nonnegative if $\delta - 1 \leq \sigma_T$, where

$$\sigma_T = \frac{\int_{\overline{M}_T}|\overline{\nabla}v_T|^2}{2(1-CT^{-1})\sum_{i=1}^{n}\sqrt{\frac{\pi}{|\partial_i\overline{M}_T|}}\int_{\partial_i\overline{M}_T}v_T^2},$$

so that $\eta \geq \delta^{-1}(\delta-1)(1-CT^{-1})/2$ for all such $\delta$. In particular by choosing $\delta = 1+\sigma_T$ we conclude that

$$Q(v_T) \geq \frac{\sigma_T(1-CT^{-1})}{2(1+\sigma_T)}\sum_{i=1}^{n}\sqrt{\pi|\partial_i\overline{M}_T|}. \tag{4.3}$$

Furthermore combining (4.1), (4.2), and (4.3) produces

$$\mathrm{E}(g) \geq \frac{\sigma_T(1-CT^{-1})}{2(1+\sigma_T)}\sum_{i=1}^{n}\sqrt{\frac{|\partial_i\overline{M}_T|}{\pi}}. \tag{4.4}$$

The desired inequality of Theorem 1.2 may be obtained from (4.4) by letting $T \to \infty$. To see this we observe that (4.1), (4.2), and (3.2) together show that the sequence of functions $\{u_T\}$ is uniformly bounded in $W^{1,2}_{loc}(\overline{M})$. Thus with the help of elliptic estimates and Sobolev embeddings, this sequence converges on compact subsets to a smooth uniformly bounded solution $u_\infty$ of

$$\overline{\Delta}u_\infty - \frac{1}{8}\overline{R}u_\infty = 0 \quad \text{on} \quad \overline{M}, \qquad u_\infty = 1 + \frac{A_\infty}{|x|} + O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty.$$

However since $\overline{M}$ approximates a cylinder on regions where it blows-up, comparison with a bounded solution of the same equation on the cylinder (as is done in [13]) shows that $u_\infty(x) \to 0$ as $x \to \partial M$; in fact the decay rate is of exponential strength. Therefore (with a bit more effort) $\sigma_T \to \sigma_\infty \geq \sigma$ and $|\partial_i\overline{M}_T| \to |\partial_i M|$, $1 \leq i \leq n$, as $T \to \infty$. This completes the proof of Theorem 1.2.

Lastly we analyze what happens when equality occurs in Theorem 1.2. By slightly modifying the arguments of this section in this special case, we find that

$$\int_{\overline{M}}|\overline{\nabla}u_\infty|^2 = 0,$$

and therefore $u_\infty$ must be constant. However this is impossible since

$$u_\infty(x) \to \begin{cases} 1 & \text{as } |x| \to \infty, \\ 0 & \text{as } x \to \partial M. \end{cases}$$

We conclude that the case of equality cannot occur.

## References


1. Andersson, L., Metzger, J.: *The area of horizons and the trapped region*. http://arxiv.org/abs/0708.4252v3[gr-qc], 2007
2. Bray, H.: Proof of the Riemannian Penrose conjecture using the positive mass theorem. J. Diff. Geom. **59**, 177–267 (2001)
3. Bray, H., Khuri, M.: PDE's which imply the Penrose conjecture. In preparation, 2008
4. Eichmair, M.: *The plateau problem for apparent horizons*. http://arxiv.org/abs/0711.4139v2[math.DG], 2007
5. Galloway, G.: Rigidity of marginally trapped surfaces and the topology of black holes. Commun. Anal. Geom. **16**, 217–229 (2008)
6. Galloway, G., Schoen, R.: A generalization of Hawking's black hole topology theorem to higher dimensions. Commun. Math. Phys. **266**(2), 571–576 (2006)
7. Herzlich, M.: A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds. Commun. Math. Phys. **188**, 121–133 (1997)
8. Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Rieman-nian Penrose inequality. J. Diff. Geom. **59**, 353–437 (2001)
9. Jang, P.-S.: On the positivity of energy in general relativity. J. Math. Phys. **19**, 1152–1155 (1978)
10. Malec, E., Ó Murchadha, N.: The Jang equation, apparent horizons, and the Penrose inequality. Class. Q. Grav. **21**, 5777–5787 (2004)
11. Metzger, J.: *Blowup of Jang's equation at outermost marginally trapped surfaces*. http://arxiv.org/abs/0711.4753v1[gr-qc], 2007
12. Penrose, R.: Naked singularities. Ann. N. Y. Acad. Sci. **224**, 125–134 (1973)
13. Schoen, R., Yau, S.-T.: Proof of the positive mass theorem II. Commun. Math. Phys. **79**(2), 231–260 (1981)
14. Yau, S.-T.: Geometry of three manifolds and existence of black hole due to boundary effect. Adv. Theor. Math. Phys. **5**(4), 755–767 (2001)


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