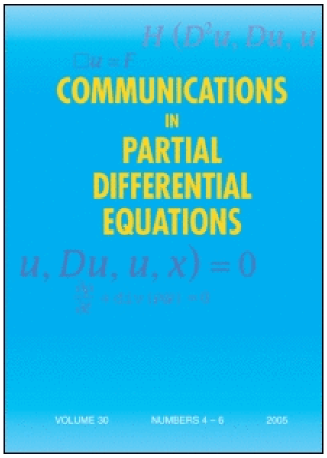

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Counterexamples to the Local Solvability of Monge-Ampère Equations in the Plane

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In this paper, we present C^∞ examples of degenerate hyperbolic and mixed type Monge-Ampère equations in the plane, which do not admit a local C^3 solution.

Keywords Local solvability; Monge-Ampère equations.

Mathematics Subject Classification Primary 35M10; Secondary 35L80.

0. Introduction

Consider the class of two-dimensional Monge-Ampère equations:

$$(u_{xx} + a(p, u, \nabla u))(u_{yy} + c(p, u, \nabla u)) - (u_{xy} + b(p, u, \nabla u))^2 = f(p, u, \nabla u), \tag{0.1}$$

where $p = (x, y)$. The question of local solvability is to ask, given smooth functions a, b, c, and f defined in a neighborhood of a point, say $(x, y) = 0$, does there always exist a C^2 function $u(x, y)$, defined in a possibly smaller domain, which satisfies (0.1)? Note that we do not ask for $u(x, y)$ to satisfy any boundary/initial conditions, have higher regularity, or to be given in a predetermined domain. This is the most elementary question that one can ask of a differential equation. Yet, it is remarkable that the basic question of whether there exist any examples of local nonsolvability, has remained open for this well-studied class of equations. The purpose of this paper is to provide such examples.

We first recall the known results. Since (0.1) is elliptic if $f > 0$, hyperbolic if $f < 0$, and of mixed type if f changes sign, the manner in which f vanishes will play the primary role in the hypotheses of any result. The classical results state that a solution always exists in the case that f does not vanish at the origin or is analytic (a, b, and c are also required to be analytic); these results follow easily from standard elliptic and hyperbolic theory when f does not vanish, and from

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the Cauchy–Kovalevskaya theorem in the case that a, b, c, and f are analytic. If $f(x, y, u, \nabla u) = f_1(x, y) f_2(x, y, u, \nabla u)$ with $f_2 > 0$, then C.-S. Lin provides an affirmative answer in Lin (1985, 1986), when $f_1 \geq 0$ or when $f_1(0) = 0$ and $\nabla f_1(0) \neq 0$. When $f_1 \leq 0$ and ∇f_1 possesses a certain nondegeneracy, Han et al. (2003) show that a solution always exists. Furthermore, in Khuri (preprint a, to appear b) the author provides an affirmative answer in the case that f_1 has a nondegenerate critical point at the origin, or degenerates to arbitrary finite order along a single smooth curve (see also Han's result, 2006). Here, we shall prove the theorem.

Theorem. *There exist sign changing and nonpositive* $f \in C^\infty(\mathbb{R}^5)$, *and* $a, b, c \in C^\infty(\mathbb{R}^5)$, *such that equation* (0.1) *possesses no* C^3 *solution in any neighborhood of the origin.*

The results mentioned above stem from work on a well-known problem in geometry, namely the local isometric embedding problem for two-dimensional Riemannian manifolds. This problem is equivalent to the local solvability of the following Monge-Ampère equation:

$$\det \nabla_{ij} u = K(\det g)(1 - |\nabla_g u|^2), \tag{0.2}$$

where g is a given smooth Riemannian metric, K is its Gaussian curvature, ∇_{ij} are second covariant derivatives, and ∇_g is the gradient with respect to g. Recently, N. Nadirashvili and Y. Yuan have proposed counterexamples to the isometric embedding problem in Nadirashvili (Preprint) and Nadirashvili and Yuan (Preprint). An immediate consequence is the local nonsolvability of equation (0.2). Although this observation was not mentioned by the authors, it is quite significant since it represents the first nontrivial example of a fully nonlinear equation exhibiting the property of local nonsolvability. (Of course in the setting of linear equations, this phenomenon has received much attention through the work of Hörmander, Nirenberg, Treves, and others since its original discovery by Lewy, 1957). The main distinction between our theorem and the results of Nadirashvili and Yuan, besides the difference in equations considered, is the fact that the proof presented here is very elementary and does not rely on any geometric significance that the equation may possess (as is exhibited with equation (0.2)); as a result it is possible that the methods presented here may be generalized to other Monge-Ampère equations.

In the remainder of this section, we will partially construct the functions a, b, c, and f of the theorem, as well as reduce the proof of this theorem to the problem of showing that certain second derivatives of any solution of (0.1) must vanish along the boundary of a sequence of squares. Define sequences of disjoint open squares $\{X^n\}_{n=1}^\infty$ and $\{X_1^n\}_{n=1}^\infty$ whose sides are aligned with the x and y-axes, and such that X^n, X_1^n are centered at $q_n = (\frac{1}{n}, 0)$, $X^n \subset X_1^n$, and X^n, X_1^n have widths $\frac{1}{2n(n+1)}$, $\frac{1}{n(n+1)}$, respectively. Set $a, b, c, f \equiv 0$ in $\mathbb{R}^2 - \bigcup X_1^n$. Define

$$X = \{(x, y) \mid |x| < 1, |y| < 1\},$$

and let $\phi \in C^\infty(\overline{X})$ be such that ϕ vanishes to infinite order on ∂X, and either $\phi(p) > 0$ or $\phi(p) < 0$ for all $p \in X$; here $\overline{X}$ denotes the closure of X. We now define

a, b, c, f in X^n by $a, b, c \equiv 0$, and

$$f(p) = \gamma_n \phi(4n(n+1)(p - q_n)), \quad p \in \overline{X}^n, \tag{0.3}$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is a sequence of positive numbers that will be chosen later, with the property that $\lim_{n\to\infty} \gamma_n = 0$. In the next section, f will be defined to be nonpositive in the remaining region $\bigcup_{n=1}^{\infty}(X_1^n - X^n)$. Therefore, by choosing ϕ to be positive or negative in each X^n, we obtain the desired sign changing or nonpositive f as mentioned in the theorem.

We now reduce the proof of the theorem as mentioned above. Suppose that a local solution, $u \in C^3$, of (0.1) exists. Let $-\mathrm{v}_n$, $+\mathrm{v}_n$ represent the left and right vertical portions of ∂X^n, respectively, and let $+\mathrm{h}_n$, $-\mathrm{h}_n$ represent the top and bottom horizontal portions of ∂X^n, respectively. Now assume that

$$u_{yy}|_{\pm \mathrm{v}_n} = 0 \quad \text{and} \quad u_{xx}|_{\pm \mathrm{h}_n} = 0, \ \text{ for all } n \geq N, \tag{0.4}$$

where N is the smallest integer such that X^N is completely contained within the domain of existence of u. Let $n_0 \geq N$, and note that (0.3) and (0.4) imply that $u_{xy}|_{\pm \mathrm{v}_{n_0}} = 0$. We may now integrate by parts to obtain a contradiction,

$$0 \neq \int_{X^{n_0}} f = \int_{X^{n_0}} u_{xx}u_{yy} - u_{xy}^2 = \int_{\partial X^{n_0}} u_{xx}u_y v_2 - u_{xy}u_y v_1 = 0,$$

where (v_1, v_2) are the components of the unit outward normal to ∂X^{n_0}. Thus, our theorem is reduced to the proof of (0.4).

The outline of the paper is as follows. In Sec. 1 we complete the construction of a, b, c, and f. Furthermore, assuming that (0.4) does not hold, we find a certain integral equality that u must satisfy. In order to violate this integral equality, we construct approximate solutions to a homogeneous degenerate hyperbolic equation in Sec. 2.

1. The Integral Equality

The purpose of this section is to construct a sequence of integral equalities, valid for C^3 solutions of (0.1) in subdomains of $X_1^n - X^n$ if (0.4) is violated. However, before obtaining the integral equalities we will first complete the construction of a, b, c, and f in the regions $X_1^n - X^n$. Extend the line segments $\pm\mathrm{v}_n$, $\pm\mathrm{h}_n$ until they reach ∂X_1^n, so that we obtain four rectangles each bounded by ∂X^n, ∂X_1^n, and the extended segments $\pm\mathrm{v}_n$, $\pm\mathrm{h}_n$. Denote the rectangles to the left and right of ∂X^n by $-V_n$, $+V_n$ respectively, and denote the rectangles to the top and bottom of ∂X^n by $+H_n$, $-H_n$ respectively. We then set $a, b, c, f \equiv 0$ in $(X_1^n - X^n) - (\pm V_n \cup \pm H_n)$, define $a = a_n(x, y)uu_y$, $b = c \equiv 0$ in $\pm V_n$, and $c = c_n(x, y)uu_x$, $a = b \equiv 0$ in $\pm H_n$, for some $a_n \in C^\infty(\pm\overline{V}_n)$ and $c_n \in C^\infty(\pm\overline{H}_n)$ to be given below. Lastly, in $\pm V_n \cup \pm H_n$ we will write $f = K_n(x, y) + g_n(x, y, \nabla u)$ for nonpositive functions $K_n \in C^\infty(\overline{\pm V_n \cup \pm H_n})$, $g_n \in C^\infty(\overline{\pm V_n \cup \pm H_n} \times \mathbb{R}^2)$ also to be given below. In order to motivate the construction of f in the regions $\pm V_n \cup \pm H_n$, we will now convert (0.1) into a quasilinear equation by applying an appropriate Legendre transformation.

Let $u \in C^3$ be a local solution of (0.1), and as above let n_0 be such that $X_1^{n_0}$ is contained within the domain of existence of u. Let $p = (p_1, p_2) \in +\mathrm{v}_{n_0}$, and assume that (0.4) is violated, so that $u_{yy}(p) \neq 0$. Then we have a well-defined C^2 Legendre transformation $T:(x,y)\mapsto(\alpha,\beta)$ defined in a sufficiently small neighborhood, B_p of p, and given by

$$\alpha = x - p_1, \quad \beta = u_y(x,y).$$

It follows that u must satisfy a quasilinear equation in the new variables.

Lemma 1. *There exist constants* $\alpha_0 > 0$, $\beta_1 > \beta_* > \beta_0$, *and a rectangle* $D = (0,\alpha_0)\times(\beta_0,\beta_1)\subset T(B_p)$ *where* $T(p) = (0,\beta_*)$, *such that in* D *we have* $|u_y u_{yy}| > 0$ *and*

$$Lu := u_{\alpha\alpha} + (K_{n_0}u_\beta)_\beta - (2\beta^{-1}K_{n_0} + \beta^2 a_{n_0} + \beta u u_{yy} a_{n_0\beta})u_\beta = G_{n_0}, \tag{1.1}$$

where

$$G_{n_0} = -(g_{n_0}u_\beta)_\beta + 2\beta^{-1}g_{n_0}u_\beta.$$

Proof. If $u_y(p) = 0$, then we could instead take any point $\bar{p}\in +\mathrm{v}_{n_0}$ near p with $u_y(\bar{p})\neq 0$ and $u_{yy}(\bar{p})\neq 0$. The existence of a rectangle D in which $|u_y u_{yy}| > 0$ now follows. Moreover, if we set

$$F = f - au_{yy} + 2bu_{xy} - cu_{xx} + b^2 - ac,$$

then for any $\psi\in C_c^\infty(D)$ we have

$$\begin{aligned}
&\int_D (u_\alpha\psi_\alpha + Fu_\beta\psi_\beta)d\alpha\, d\beta \\
&= \int_{T^{-1}(D)}\left[\left(u_x - \frac{u_{xy}}{u_{yy}}u_y\right)\left(\psi_x - \frac{u_{xy}}{u_{yy}}\psi_y\right) + F\left(\frac{1}{u_{yy}}\right)^2 u_y\psi_y\right]u_{yy}dx\, dy \\
&= \int_{T^{-1}(D)}[(u_x u_{yy} - u_{xy}u_y)\psi_x + (u_y u_{xx} - u_x u_{xy})\psi_y]\, dx\, dy \\
&= -2\int_{T^{-1}(D)} F\psi\, dx\, dy \\
&= -2\int_D \beta^{-1}Fu_\beta\psi d\alpha\, d\beta.
\end{aligned}$$

Recalling that $f = K_{n_0} + g_{n_0}$ and $b = c \equiv 0$ in $+V_{n_0}$ we obtain

$$u_{\alpha\alpha} + (K_{n_0}u_\beta)_\beta - (2\beta^{-1}K_{n_0} + (au_{yy}u_\beta)_\beta u_\beta^{-1} - 2\beta^{-1}au_{yy})u_\beta = G_{n_0},$$

from which (1.1) follows with $a = a_{n_0}uu_y$. □

We may view (1.1) as a linear equation which possesses a solution $u\in C^2(D)$. Since K_{n_0} is nonpositive in $+V_{n_0}$, equation (1.1) is degenerate hyperbolic in D. Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{g}\in C^\infty(D)$ with $\bar{a}\geq 0$, and consider the linear degenerate hyperbolic equation:

$$z_{\alpha\alpha} - (\bar{a}z_\beta)_\beta + \bar{b}z_\beta + \bar{c}z_\alpha + \bar{d}z = \bar{g}. \tag{1.2}$$

The local solvability of (1.2) is highly dependent upon certain relationships between the coefficients $\bar{a}$ and $\bar{b}$, the so called Levi conditions. One of the most powerful Levi conditions was given by Oleinik (1970), who proved that the Cauchy problem for (1.2), with data prescribed on the line $\alpha = 0$, is well-posed if there exists an integer $J > 0$ and constants $A, B > 0$, $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_J$, such that either

$$B(\alpha - \alpha_{j-1})\bar{b}^2 \leq A\bar{a} + \bar{a}_\alpha \quad \text{or} \quad B(\alpha_j - \alpha)\bar{b}^2 \leq \left[A + \frac{1}{B(\alpha_j - \alpha)}\right]\bar{a} - \bar{a}_\alpha \tag{1.3}$$

holds for $\alpha_{j-1} \leq \alpha \leq \alpha_j$, $j = 1, \ldots, J$. Note that (1.3) implies that either

$$0 \leq A\bar{a} + \bar{a}_\alpha \quad \text{or} \quad 0 \leq \left[A + \frac{1}{B(\alpha_j - \alpha)}\right]\bar{a} - \bar{a}_\alpha,$$

so that we must have either $\bar{a}(\alpha, \bar{\beta}) = 0$ for all $\alpha_{j-1} \leq \alpha \leq \bar{\alpha}$, or all $\bar{\alpha} \leq \alpha \leq \alpha_j$, respectively, if $\bar{a}(\bar{\alpha}, \bar{\beta}) = 0$. Therefore, Oleinik's result does not allow the coefficient $\bar{a}$ to have an infinite sequence of isolated zeros as $\alpha \to 0$, as would be the case if $\bar{a}$ exhibited fast oscillations near $\alpha = 0$. In fact, counterexamples to the local solvability of (1.2) have been found (Egorov, 1993) in the case that $\bar{a}$ possesses this type of behavior.

With this intuition, we will define K_n in $+V_n$ to have special fast oscillations as $x \to \partial X^n$. Let $x_n = x - (\frac{1}{n} + \frac{1}{4n(n+1)})$, so that in the new coordinates $+V_n$ is given by

$$+V_n = \left\{(x_n, y) \,|\, 0 < x_n < \frac{1}{4n(n+1)}, \frac{-1}{4n(n+1)} < y < \frac{1}{4n(n+1)}\right\}.$$

Let k_n be the smallest integer such that $k_n > \frac{4n(n+1)}{\pi}$, and set $I_k = (\frac{1}{\pi(k+1)}, \frac{1}{\pi k})$, $k \in \mathbb{Z}_{>0}$. Then define a smooth function K for $x_n > 0$ by

$$K(x_n) = \begin{cases} e^{-m_k^{-2} - \sin^{-2}(\frac{1}{x_n})} & \text{if } x_n \in I_k,\ k \ \in \mathbb{Z}_{k \geq k_n}, \\ 0 & \text{if } x_n \in \partial I_k, \\ 0 & \text{if } x_n \geq \dfrac{1}{\pi k_n}, \end{cases}$$

where m_k denotes the unique zero of $\cos(\frac{1}{x_n})$ in the interval I_k. The function K will be used to give K_n fast oscillations, however K_n will also be required to have specific behavior in the y-direction as well. In order to accomplish this we partition $+V_n$ into small rectangles

$$\mathcal{R}_{k,i,n} = \{(x_n, y_n) \,|\, x_n \in I_k, ik^{-1/2} < y_n < (i+1)k^{-1/2}\}, \quad i = 0, 1, \ldots, \bar{i}(k, n),$$

where $y_n = y + \frac{1}{4n(n+1)}$ and $\bar{i}(k, n)$ denotes the largest integer such that $\mathcal{R}_{k,\bar{i}(k,n),n} \cap +V_n \neq \emptyset$. Next let $\psi \in C^\infty(-\infty, \infty)$ be a nonnegative 1-periodic function such that

$$\psi(\bar{y}) = \begin{cases} 1 & \text{if } \dfrac{1}{4} \leq \bar{y} \leq \dfrac{3}{4}, \\ 0 & \text{if } 0 \leq \bar{y} \leq \dfrac{1}{8} \text{ or } \dfrac{7}{8} \leq \bar{y} \leq 1, \end{cases}$$

and set

$$\Psi_n(x_n, y_n) = \begin{cases} \psi(k^{1/2}y_n) & \text{if } (x_n, y_n) \in \mathcal{R}_{k,i,n}, \ 0 \le i < \bar{i}(k,n), \\ 0 & \text{if } (x_n, y_n) \in \mathcal{R}_{k,\bar{i}(k,n),n}. \end{cases}$$

We then define

$$K_n(x, y) = -\gamma_n K(x_n)\Psi_n(x_n, y_n), \quad (x, y) \in +V_n,$$

where γ_n was given in (0.3). Note that $K_n \le 0$ in $+V_n$ and $K_n \in C^\infty(+\overline{V}_n)$. Furthermore, we define K_n analogously in the rectangles $-V_n$, $\pm H_n$, where in $\pm H_n$ the roles of x and y are reversed.

We now choose a_n, c_n, and g_n. Set $\xi_1(x_n) = \sin^{-4}(\frac{1}{x_n})$ and define

$$a_n(x, y) = 2\xi_1'(x_n)\sqrt{\gamma_n K(x_n)}\Psi_n(x_n, y_n), \quad (x, y) \in +V_n.$$

Note that $a_n \in C^\infty(+\overline{V}_n)$. We also define a_n in $-V_n$ and c_n in $\pm H_n$ similarly, where in $\pm H_n$ the roles of x and y are reversed. Next let $\{N_k\}_{k=1}^\infty$ be a sequence of positive integers, and let $\psi_{N_k} \in C^\infty(-\infty, \infty)$ be a nonnegative 1-periodic function such that

$$\int_0^1 \psi_{N_k}'(\bar{y})\psi^{(N_k)}(\bar{y})d\bar{y} \ge 1, \quad k \in \mathbb{Z}_{>0}; \tag{1.4}$$

here and below we use the notation $\psi^{(j)}(\bar{y}) = \frac{d^j}{d\bar{y}^j}\psi(\bar{y})$. The integers N_k are to be chosen converging to infinity sufficiently slow so that

$$\lim_{k\to\infty} e^{-\frac{1}{2}m_k^{-2}} \sup_{\bar{y}\in(0,1)} \left|\psi_{N_k}^{(j)}(\bar{y})\right| = 0, \quad \text{for each } j \in \mathbb{Z}_{\ge 0}. \tag{1.5}$$

If we did not require ψ_{N_k} to be nonnegative, that is, if in the theorem we only wished to construct examples of mixed type Monge-Ampère equations (see the definition of g_n below), then we could simply take $\psi_{N_k} = \psi^{(N_k-1)}$, since the Poincaré inequality implies that

$$\int_0^1 \psi^2(\bar{s})d\bar{s} \le \pi^{-2N_k}\int_0^1 [\psi^{(N_k)}]^2(\bar{s})d\bar{s},$$

where we have used the fact that π^2 is the principal eigenvalue of $-\frac{d^2}{d\bar{s}^2}$ on the interval $(0, 1)$. Let $\{\tau_k\}_{k=1}^\infty$ be a sequence of positive numbers to be chosen later with $\lim_{k\to\infty}\tau_k = \infty$, then for $(x, y) \in +V_n$ we define

$$g_n(x, y, \nabla u) = \begin{cases} \tau_k^{-N_k} K_n(x, y)\psi_{N_k}(\tau_k s_n(x, y, u_y)) & \text{if } (x_n, y_n) \in \mathcal{R}_{k,i,n}, \ 0 \le i < \bar{i}(k,n), \\ 0 & \text{if } (x_n, y_n) \in \mathcal{R}_{k,\bar{i}(k,n),n}, \end{cases}$$

where $s_n = u_y(x, y) + \int_0^{x_n}\sqrt{\gamma_n K(\bar{x})}d\bar{x}$. Note that $g_n \le 0$ and according to (1.5), $g_n \in C^\infty(+\overline{V}_n \times \mathbb{R}^2)$. In the regions $-V_n$, $\pm H_n$, g_n is defined similarly, where again the roles of x and y are reversed in $\pm H_n$. Lastly, by choosing a sequence $\{\gamma_n\}_{n=1}^\infty$ which converges to zero sufficiently fast we have $a, b, c, f \in C^\infty(\mathbb{R}^5)$.

Now that a, b, c, and f are fully defined, we shall obtain a sequence of integral equalities for u inside $+V_{n_0}$. In order to construct the domains in which these integral equalities will be valid, we need the following change of coordinates $T_1 : (\alpha, \beta) \mapsto (t, s)$ given by

$$t = \alpha, \quad s(\alpha, \beta) = \beta + \int_0^{\alpha} \sqrt{\gamma_{n_0} K(\bar{\alpha})} d\bar{\alpha},$$

which is well-defined near $T(p)$. Let $\{\mathscr{R}_{k,i_k,n_0}\}_{k=1}^{\infty}$ be a sequence of rectangles such that $\mathscr{R}_{k,i_k,n_0} \cap \{(x, y) \in +V_{n_0} \mid y = p_2\} \neq \emptyset$, so that this sequence converges to p. Then for $(x, y) \in \mathscr{R}_{k,i_k,n_0}$ we have $|x - p_1| + |y - p_2|^2 = O(k^{-1})$, which implies that

$$s = u_{yy}(p)y + (u_y(p) - u_{yy}(p)p_2) + O(k^{-1}).$$

Therefore if $\tau_k \geq k$ and k is large enough, there exist integers J_k such that the rectangles in the ts-plane,

$$R_{k,\tau_k} = \{(t, s) \mid t \in I_k, J_k < \tau_k s < J_k + 1\},$$

satisfy

$$T^{-1}(T_1^{-1}(R_{k,\tau_k})) \subset \{(x, y) \in +V_{n_0} \mid \Psi_{n_0}(x_{n_0}, y_{n_0}) \equiv 1\} \cap \mathscr{R}_{k,i_k,n_0}.$$

It follows that $a_{n_0\beta}(x, y) = (a_{n_0 y} u_{yy}^{-1})(x, y) = 0$ for $(x, y) \in T^{-1}(T_1^{-1}(R_{k,\tau_k}))$. Then in R_{k,τ_k} the expression for the operator L of (1.1) becomes

$$Lu = u_{tt} + 2Au_{ts} + Bu_s,$$

where the coefficients A and B are smooth in ts-coordinates and are given by

$$A = \sqrt{\gamma_{n_0} K}, \quad B = 2\beta^{-1}\gamma_{n_0} K - 2\beta^2 \xi_1' \sqrt{\gamma_{n_0} K} + (\sqrt{\gamma_{n_0} K})_t.$$

Let $z_k \in C^{\infty}(\overline{R}_{k,\tau_k})$ vanish identically on the boundary, then multiplying equation (1.1) through by z_k and integrating by parts gives the desired integral equality

$$\int_{R_{k,\tau_k}} uL^* z_k dt\, ds = \int_{R_{k,\tau_k}} G_{n_0} z_k dt\, ds, \tag{1.6}$$

where L^* denotes the formal adjoint of L.

2. Violation of the Integral Equality

In this section we will violate the integral equality (1.6) for large k, by choosing the sequence $\{\tau_k\}_{k=1}^{\infty}$ to grow sufficiently fast, and by constructing approximate solutions of the homogeneous adjoint equation $L^* z = 0$ vanishing identically on $\partial R_{k,\tau_k}$, so that the left-hand side of (1.6) tends to zero much faster than the right-hand side.

The approximate solutions will be of the form

$$z_k = e^{\eta(t,s)} \sum_{i=0}^{N_k} \tau_k^{-i} a_i(t, s) b_i(\tau_k s) := e^{\eta} \bar{z},$$

where $a_i, \eta \in C^\infty(R_{k,\tau_k})$ and $b_i = \psi^{(N_k-i)}$. A calculation shows that

$$e^{-\eta}L^* z_k = \bar{z}_{tt} + 2A\bar{z}_{ts} + \overline{B}\bar{z}_s + \overline{C}\bar{z}_t + \overline{D}\bar{z}, \tag{2.1}$$

where

$$\overline{B} = 2A_t - B + 2A\eta_t, \quad \overline{C} = 2(A_s + A\eta_s + \eta_t), \quad \overline{D} = \eta_t^2 + \eta_{tt} + \overline{B}\eta_s + \overline{B}_s.$$

In order to simplify (2.1) we choose η so that $\overline{B} \equiv 0$, that is we set

$$\eta(t,s) = \int_{m_k}^{t} (-\beta^2 \xi_1'(\bar{t}) + \beta^{-1}\sqrt{\gamma_{n_0} K(\bar{t})})d\bar{t} + \frac{1}{4}\xi_2(t),$$

where $\xi_1(t) = \sin^{-4}(\frac{1}{t})$ and $\xi_2(t) = \sin^{-2}(\frac{1}{t})$. It follows that

$$C^{-1}e^{-\sigma_1 \sin^{-4}(\frac{1}{t})} \leq e^{\eta} \leq Ce^{-\sigma_2 \sin^{-4}(\frac{1}{t})} \quad \text{in } R_{k,\tau_k}, \tag{2.2}$$

for some constants $C, \sigma_1, \sigma_2 > 0$ independent of k. Furthermore, (2.1) becomes

$$\begin{aligned} e^{-\eta}L^* z_k &= 2\tau_k A a_{0t} b_0' + \sum_{i=0}^{N_k-1} \tau_k^{-i}(a_{itt} + 2Aa_{its} + \overline{C}a_{it} + \overline{D}a_i + 2Aa_{(i+1)t})b_i \\ &\quad + \tau_k^{-N_k}(a_{N_k tt} + 2Aa_{N_k ts} + \overline{C}a_{N_k t} + \overline{D}a_{N_k})b_{N_k}. \end{aligned}$$

This suggests that we inductively choose $a_0 = \operatorname{sgn}(u_y u_{yy}^{-1}(p)) := \pm 1$,

$$2Aa_{(i+1)t} = -a_{itt} - 2Aa_{its} - \overline{C}a_{it} - \overline{D}a_i, \quad a_{i+1}(m_k, s) = 0, \ \ i = 0, \ldots, N_k - 1.$$

Therefore (2.2) shows that $z_k \in C^\infty(\overline{R}_{k,\tau_k})$ vanishes identically on $\partial R_{k,\tau_k}$, and

$$\max_{j=0,1,2} \sup_{(t,s)\in R_{k,\tau_k}} e^{\eta(t,s)}|\nabla^j a_i(t,s)|(1 + |\overline{C}(t,s)| + |\overline{D}(t,s)|) \leq M(i,k,u) \tag{2.3}$$

for some constants $M(i,k,u)$ which only depend on τ_k through their dependence on u. As a result, it is clear that τ_k can be chosen so that for any $w \in C^3(+\overline{V}_{n_0})$ with $w_y(p) \neq 0$ and $w_{yy}(p) \neq 0$,

$$C_1(\psi_{N_k}) \sum_{i=1}^{N_k} \tau_k^{-i} C_{N_k - i}(\psi) M(i,k,w) \leq \overline{M}_0(w)e^{-k^4}, \tag{2.4}$$

$$\tau_k^{-1} M(N_k, k, w) \leq \overline{M}_1(w)e^{-k^4}, \quad \tau_k^{-1} C_{N_k}(\psi) C_0(\psi_{N_k}) \leq k^{-4}, \tag{2.5}$$

where

$$C_j(\psi) := \sup_{\bar{s}\in(0,1)} |\psi^{(j)}(\bar{s})|, \quad C_j(\psi_{N_k}) := \sup_{\bar{s}\in(0,1)} |\psi_{N_k}^{(j)}(\bar{s})|,$$

and the constants $\overline{M}_0(w)$ and $\overline{M}_1(w)$ depend only on $\beta_0(w)$ and $\beta_1(w)$ (see Lemma 1). Then combining (2.3) and (2.5) we have

$$\left|\int_{R_{k,\tau_k}} uL^* z_k dt\, ds\right| \leq C\tau_k^{-N_k-1} M(N_k, k, u) \leq C\overline{M}_1(u) e^{-k^4} \tau_k^{-N_k}. \tag{2.6}$$

We now estimate the right-hand side of the integral equality (1.6). First observe that Lemma 1 together with the definition of g_{n_0} yields

$$G_{n_0}(t, s) = \gamma_{n_0} \tau_k^{-N_k+1} \psi'_{N_k}(\tau_k s) K(t) u_y u_{yy}^{-1} + O(\tau_k^{-N_k} C_0(\psi_{N_k}) e^{-m_k^{-2}}), \quad (t, s) \in R_{k,\tau_k}.$$

Moreover, using (2.3) we have

$$z_k(t, s) = \operatorname{sgn}(u_y u_{yy}^{-1}(p)) e^{\eta} \psi^{(N_k)}(\tau_k s) + O\bigg(\sum_{i=1}^{N_k} \tau_k^{-i} C_{N_k-i}(\psi) M(i, k, u)\bigg).$$

We now apply (1.4), (2.2), (2.4), and (2.5) to obtain

$$\begin{aligned}
&\int_{R_{k,\tau}} G_{n_0} z_k dt\, ds \qquad (2.7)\\
&\geq Ce^{-m_k^{-2}} \tau_k^{-N_k} \bigg(\int_{\frac{1}{\pi(k+1)}}^{\frac{1}{\pi k}} e^{-\sigma_1 \sin^{-4}(\frac{1}{t}) - \sin^{-2}(\frac{1}{t})} dt\bigg)\bigg(\int_0^1 \psi'_{N_k}(\bar{s}) \psi^{(N_k)}(\bar{s}) d\bar{s}\bigg)\\
&\quad + O\bigg(\tau_k^{-N_k-1} e^{-m_k^{-2}} C_{N_k}(\psi) C_0(\psi_{N_k}) + \tau_k^{-N_k} e^{-m_k^{-2}} C_1(\psi_{N_k}) \sum_{i=1}^{N_k} \tau_k^{-i} C_{N_k-i}(\psi) M(i, k, u)\bigg)\\
&\geq C_1 k^{-2} e^{-m_k^{-2}} \tau_k^{-N_k} + O\big(\tau_k^{-N_k} e^{-m_k^{-2}} \overline{M}_0(u) e^{-k^4}\big),
\end{aligned}$$

for some constants $C, C_1 > 0$ independent of k.

We may now complete the proof of the theorem. Combining (1.6), (2.6), and (2.7) produces

$$C^{-1} k^{-2} e^{-m_k^{-2}} \tau_k^{-N_k} \leq \int_{R_{k,\tau_k}} G_{n_0} z_k dt\, ds = \int_{R_{k,\tau_k}} uL^* z_k dt\, ds \leq C\overline{M}_1(u) e^{-k^4} \tau_k^{-N_k},$$

which leads to a contradiction for large k. It follows that $u_{yy}(p) = 0$. Since $p \in +\mathrm{v}_{n_0}$ was arbitrary we must then have $u_{yy}|_{+\mathrm{v}_{n_0}} = 0$. Moreover, the definitions of a, b, c, and f in the regions $-V_{n_0}$, $\pm H_{n_0}$ is such that the same arguments used in Secs. 1 and 2 may be applied to show that

$$u_{yy}|_{-\mathrm{v}_{n_0}} = 0, \quad u_{xx}|_{\pm \mathrm{h}_{n_0}} = 0.$$

Lastly, since X^{n_0} was chosen arbitrarily inside the domain of existence of u, (0.4) is valid. The theorem now follows from the arguments at the end of the introduction.

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