Homework 12

1. Chapter 17, Problem 5.

2. Chapter 18, Problem 4.

3. Chapter 18, Problem 5.

4. Chapter 18, Problem 6.

5. Compute the de Rham groups of the n-dimensional torus: $H^k_{dR}(T^n)$.

6. Let M and N be compact smooth manifolds without boundary. A differential form ω on $M \times N$ is called *decomposable* if

$$\omega = \pi_M^* \alpha \wedge \pi_N^* \beta$$

for some forms α and β on M and N, respectively; here π_M and π_N are the canonical projections.
a) Show that

$$\Delta_{M \times N}(\pi_M^* \alpha \wedge \pi_N^* \beta) = \pi_M^* \Delta_M \alpha \wedge \pi_N^* \beta + \pi_M^* \alpha \wedge \pi_N^* \Delta_N \beta,$$

where Δ denotes Hodge Laplacian.
b) Show that the $\mathbb{R}$-span of the decomposable forms on $M \times N$ is L^2-dense in $\Lambda^*(M \times N)$.
c) Conclude that

$$\mathcal{H}^*(M \times N) \cong \mathcal{H}^*(M) \otimes \mathcal{H}^*(N), \qquad \pi_M^* \alpha \wedge \pi_N^* \beta \leftrightarrow \alpha \otimes \beta.$$

d) Conclude that

$$H^k_{dR}(M \times N) \cong \bigoplus_{p+q=k} H^p_{dR}(M) \otimes H^q_{dR}(N).$$

This is the Künneth Formula for compact manifolds.

7. Let M and N be as in problem 6 above. Prove the following product formula for the Euler characteristic

$$\chi(M \times N) = \chi(M)\chi(N).$$