Name: ______________________ Id: ______________________

1. (10 points) Find the solution $y(x)$ of the intial value problem

$$\begin{aligned} \frac{dy}{dx} &= y^2 \sin x \\ y(0) &= 1. \end{aligned}$$

EQUATION IS SEPARABLE, SO

$$\int \frac{dy}{y^2} = \int \sin x \, dx$$

$$-\frac{1}{y} = -\cos x + c$$

$$y = \frac{1}{\cos x - c}$$

SINCE $y(0) = 1 = \frac{1}{\cos(0) - c} = \frac{1}{1-c}$, $c = 0$.

$$\therefore \; y = \frac{1}{\cos x} = \sec x.$$

CHECK:

$$\frac{dy}{dx} = \frac{-1}{(\cos x)^2}(-\sin x) = \left(\frac{1}{\cos x}\right)^2 \sin x = y^2 \sin x$$

YEP.

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2. After 3 days a sample of radon-222 decayed to 60% of its original amount.

(a) (10 points) What is the half-life of radon-222?

SINCE DECAY OF RADIOACTIVE ELEMENTS IS EXPONENTIAL, WE HAVE

$R(t) = Ae^{-kt}$ FOR SOME A AND k. WE CAN TAKE $A=1$, SINCE WE ARE DEALING W/ PERCENTAGES.

WE KNOW $R(3) = .6 = e^{-3k}$

SO $\ln(.6) = -3k$, SO $k = \frac{-\ln(.6)}{3}$

THE HALF-LIFE IS THE VALUE OF t SO THAT

$R(t) = .5$.

$.5 = e^{\ln(.6)/3 \, t}$ (THIS IS OK, SINCE $\ln(.6) < 0$)

$\ln(.5) = \frac{\ln(.6)}{3} t$

$t = \frac{3\ln(.5)}{\ln(.6)}$ (SINCE $\frac{\ln(.5)}{\ln(.6)}$ IS A LITTLE BIGGER THAN 1, THIS IS REASONABLE)

(b) (10 points) How long would it take the sample to decay to 10% of its original amount?

HERE WE JUST SOLVE FOR t WHEN.

$.1 = e^{\frac{\ln(.6)}{3} t}$

$\ln(.1) = \frac{\ln(.5)}{3} t$

SO

$t = \frac{3\ln(.1)}{\ln(.5)}$

3. Determine whether the following sequences are convergent or not. If convergent compute their limits. Show your work!

(a) (10 points)

$$a_n = \frac{9^{n+1}}{10^n} = 9 \cdot \left(\frac{9}{10}\right)^n$$

CONVERGES TO 0, SINCE

$$\lim_{n \to \infty} \left(9 \left(\frac{9}{10}\right)^n \right) = 9 \lim_{n \to \infty} \left(\frac{9}{10}\right)^n = 9 \cdot 0 = 0$$

(b) (10 points)

$$a_n = \frac{n^3}{n^2+1}$$

DIVERGES TO $+\infty$, SINCE

$$\lim_{n \to \infty} \frac{n^3}{n^2+1} = \lim_{n \to \infty} \frac{n^3/n^2}{n^2/n^2 + 1/n^2} = \lim_{n \to \infty} \frac{n}{1 + 1/n^2} = \infty$$

(c) (10 points)

$$a_n = \ln(n+2) - \ln(n+1)$$

SINCE $\ln(n+2) - \ln(n+1) = \ln\left(\frac{n+2}{n+1}\right)$

SEQUENCE CONVERGES TO 0, SINCE

$$\lim_{n \to \infty} \ln\left(\frac{n+2}{n+1}\right) = \ln\left(\lim_{n \to \infty} \frac{n+2}{n+1}\right) = \ln 1 = 0$$

4. (a) (10 points) Does the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} = \frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \cdots$$

converge or diverge? Explain why.

USE THE INTEGRAL TEST.

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} \qquad u = \ln x \quad x=2 \Rightarrow u = \ln 2$$
$$du = \frac{dx}{x} \quad x = \infty \Rightarrow u = \infty$$

$$= \int_{\ln 2}^{\infty} \frac{du}{u^2}$$

$$= -\frac{1}{u}\Big|_{\ln 2}^{\infty} = 0 + \frac{1}{\ln 2}$$

SINCE THE INTEGRAL IS FINITE, THE SERIES CONVERGES.

(b) (15 points) Does

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sqrt{n}}{n+1} = \frac{1}{2} - \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} - \frac{2}{5} + \cdots$$

converge or diverge? Is the series absolutely convergent? Explain why.

SINCE THIS IS AN ALTERNATING SERIES, WE ONLY NEED CHECK THAT $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1}$ IS ZERO.

BUT IT IS, SINCE $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = \lim_{n\to\infty} \frac{1}{\sqrt{n} + \frac{1}{\sqrt{n}}} = 0$.

SO THE SERIES IS CONVERGENT.

IT IS NOT ABSOLUTELY CONVERGENT, HOWEVER, SINCE $\sum_1^{\infty} \frac{\sqrt{n}}{n+1}$ DIVERGES. WE CAN COMPARE THIS TO $\sum_1^{\infty} \frac{1}{\sqrt{n}}$ USING THE LIMIT COMPARISON TEST:

$$\lim_{n\to\infty} \left(\frac{\sqrt{n}/n+1}{1/\sqrt{n}} \right) = \lim_{n\to\infty} \left(\frac{n}{n+1} \right) = 1.$$

SINCE $\sum_1^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES $\begin{pmatrix} p\text{-SERIES} \\ p = 1/2 \end{pmatrix}$

SO DOES $\sum_1^{\infty} \frac{\sqrt{n}}{n+1}$.

(c) (10 points) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^n}$$

Please show your work and explain your reasoning!

$$\frac{2^{2n+1}}{5^n} = 2 \cdot \frac{2^{2n}}{5^n} = 2 \cdot \left(\frac{4}{5}\right)^n .$$

SO THE SERIES IS A GEOMETRIC SERIES W/ RATIO $\frac{4}{5}$.

$$\sum_{n=0}^{\infty} \frac{2^{2n+1}}{5^n} = 2\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n = 2\left(\frac{1}{1-4/5}\right) = 2(5) = 10 .$$

(d) (15 points) Does the series

$$\sum_{n=1}^{\infty} \frac{5^{2n}}{n^2 9^n} = \sum_{n=1}^{\infty} \frac{25^n}{n^2 9^n}$$

converge of diverge? Explain why.

WE CAN USE THE RATIO TEST:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(\frac{25^{n+1}}{(n+1)^2 9^{n+1}} \cdot \frac{n^2 \cdot 9^n}{25^n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{25}{9} \frac{n^2}{(n+1)^2}\right) = \frac{25}{9}$$

SINCE THE RATIO IS >1, THE SERIES DIVERGES.

5. (20 points) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n$$

Please show your work and explain your reasoning!

USING THE RATIO TEST, WE HAVE

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n(x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{1}{2}\right||x-1| = \frac{1}{2}|x-1|$$

THIS MEANS THE SERIES CONVERGES WHEN

$$\frac{1}{2}|x-1| < 1$$

ie $|x-1| < 2$

ie $-2 < x-1 < 2$

so $-1 < x < 3$

RADIUS OF CONVERGENCE IS 2 (CENTER IS 1)

WE HAVE TO CHECK $x=-1$ AND $x=3$.

IF $x=3$, SERIES BECOMES $\sum_{n=0}^{\infty}\frac{n}{2^n}2^n = \sum_{n=0}^{\infty} n$ DIVERGES.

IF $x=-1$, SERIES IS $\sum\frac{n}{2^n}(-2)^n = \sum_{n=0}^{\infty}(-1)^n n$, DIVERGES.

SO INTERVAL OF CONVERGENCE IS

$(-1, 3)$.

6. (a) (20 points) Expand the function $f(x) = \frac{x}{(1-x)^2}$ as a power series nearby $x = 0$; in other words compute the MacLaurin series for $f(x)$. What is its radius of convergence?

SINCE $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

TAKE DERIV $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots$

MULTIPLY BY X,

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + \cdots$$

SINCE THE RADIUS OF CONVERGENCE OF $\sum_{n=0}^{\infty} x^n$ IS 1, SO IS THE RADIUS FOR $\sum_{n=1}^{\infty} n x^n$.

(b) (20 points) Compute the sum of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

How many terms of the series does one need to add/substract in order to compute the sum correct to one decimal place?

FROM PART (a) $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$. THIS SERIES IS JUST $x = \frac{-1}{2}$, SO THE SUM IS $\frac{-1/2}{(1+1/2)^2} = \frac{-1/2}{9/4} = \frac{-2}{9}$

TO GET THE SUM TO ONE PLACE, WE NEED

$$\frac{n+1}{2^{n+1}} < 0.05$$

$n = 5$ FAILS, BUT $n = 6$ WORKS.

SO WE NEED 6 TERMS : $\sum_{n=1}^{6} n \left(\frac{-1}{2}\right)^n$

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7. (10 points) Use Taylor/MacLaurin series to evaluate the following limit

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$\frac{1-\cos x}{x^2} = \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots\right)}{x^2}$$

$$= \frac{x^2/2 - x^4/4! + \cdots}{x^2}$$

$$= \frac{1}{2} - \frac{x^2}{4!} + \cdots$$

So

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{2}-\frac{x^2}{4!}+\cdots\right) = \frac{1}{2}.$$

8. (a) (15 points) Use power series to solve the initial value problem

$$y' = x^2 y, \qquad y(0) = 1$$

WRITE y AS A POWER SERIES, SO

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n + \cdots \qquad y(0)=1 \Rightarrow C_0 = 1.$$

$$y' = C_1 + 2C_2x + 3C_3x^2 + \cdots + nC_nx^{n-1} + (n+1)C_{n+1}x^n + \cdots$$

$$x^2y^* = C_0x^2 + C_1x^3 + \cdots + C_{n-2}x^n + \cdots$$

SINCE $y' = x^2y$,

$C_0 = 1$

$C_1 = 0$

$C_2 = 0$

$3C_3 = 1$, so $C_3 = \frac{1}{3}$

$4C_4 = C_1 = 0$

$5C_5 = C_2 = 0$

$6C_6 = C_3 = \frac{1}{3}$, so $C_6 = \frac{1}{18}$

IN GENERAL

$nC_n = C_{n-3}$

SO

$C_n = \frac{1}{n}C_{n-3}$

$C_n = \frac{1}{n(n-3)(n-6)} \cdots$ IF n IS A MULT. OF 3,

0 OTHERWISE

SO,

$$y = 1 + \frac{x^3}{3} + \frac{x^6}{18} + \frac{x^9}{9\cdot 6\cdot 3} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{\left(\frac{x}{3}\right)^{3n}}{n!}$$

(b) (5 points) What is the radius of convergence of the solution? Can you express the power series solution in terms of known elementary functions?

TO GET THE RADIUS OF CONV, LOOK AT

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x}{3}\right)^{3n+3}}{(n+1)!} \cdot \frac{n!}{\left(x/3\right)^{3n}} \right| = \lim_{n\to\infty} \left| \frac{x^3}{3^3(n+1)} \right| = 0.$$

SO THE RADIUS OF CONVERGENCE IS ∞.

THE SERIES IS THE MACLAURIN SERIES FOR $e^{x^3/3}$

(WE COULD GET THIS DIRECTLY SINCE THE D.E. IS SEPARABLE)