Spacetime Bartnik Mass Positivity and Temporal Monotonicity for Black Holes

Lars Andersson[1], Marcus Khuri[2], Marc Mars[3], and Walter Simon[4]

[1]Kungliga Tekniska högskolan,
100 44 Stockholm, Sweden
[2]Department of Mathematics, Stony Brook University,
Stony Brook, NY 11794, USA,
marcus.khuri@stonybrook.edu
[3]Institute of Fundamental Physics and Mathematics,
Universidad de Salamanca, Plaza de la Merced s/n,
E-37008 Salamanca, Spain,
marc@usal.es
[4]Fakultät für Physik, Universität Wien,
Boltzmanngasse 5, 1090 Wien, Austria,
walter.simon@univie.ac.at

Abstract

We define a quasilocal mass of Bartnik type, and establish its positivity and temporal monotonicity properties for two classes of domains associated with black holes. More precisely, we first show that the quasilocal mass is strictly positive for spacelike hypersurfaces that are: compact with apparent horizon boundary or noncompact with asymptotically flat ends and containing an apparent horizon in any admissible extension. Secondly, we show that the quasilocal mass is monotonically nondecreasing in time within evolutionary scenarios related to the two aforementioned settings.

1 Introduction

1.1 Motivation

Gravitational energy is non-local and hence it is an interesting problem to find useful quasilocal notions of mass or energy-momentum [27]. Given a spacelike hypersurface Ω in spacetime, its quasilocal mass should measure the total mass or energy-momentum contained in Ω. There are many competing definitions of quasilocal masses with interesting properties [31].

Here we consider a quasilocal mass $m(\Omega)$ of Bartnik type [11, 12, 13, 25]. We recall that Bartnik's idea is to define 'admissible' extensions of Ω which are asymptotically flat, and satisfy an energy condition such that the ADM masses of the extensions are well defined and nonnegative. A key issue is to find a suitable 'no-horizons' condition, that should render the infimum of the ADM masses at a designated end of all admissible extensions nonnegative, and positive unless the original domain Ω arises from Minkowski space. This problem can be studied in the Riemannian setting, where the no-horizons condition is expressed in terms of the absence of minimal surfaces, or in the general case of initial data sets where one might expect apparent horizons to play a role. While the Riemannian case has been studied extensively and is fairly well understood, there is very little known about the general setting involving initial data. This discrepancy is in part due to the different status of the Penrose inequality in the respective settings.

In this work we are on the one hand concerned with positivity proofs for $m(\Omega)$, (cf Theorems 1 and 2), as applications of the recently obtained Penrose-like inequality [1, Theorem 1.1]. On the other hand, in a spacetime foliated by spacelike hypersurfaces, we prove short-time monotonicity of $m(\Omega)$ along marginally outer trapped tubes as well as for generic achronal surfaces. The former setting (Theorem 3) fits the intuitive picture of a 'black hole' which swallows energy upon time evolution. It may be compared with, although distinguished from, the well-known area laws, which state that the areas of event horizons [20, 15] and dynamical horizons [9], do not decrease to the future. On the other hand, our second monotonicity result (Theorem 4) seems much less intuitive, as the achronal boundary is not specified. However, the topologcal requirements for this result are such that apparent horizons are necessarily present as well, if only in any admissible extensions.

We anticipate here that in the positivity Theorems 1 and 2 we admit data which satisfy the dominant energy condition, while we restrict ourselves to vacuum domains in Theorems 3 and 4. Nevertheless, for technical reasons we need to admit matter fields in the admissible extensions of the monotonicity results as well.

1.2 Standard Definitions

Throughout the paper all manifolds are assumed to be smooth, connected and orientable unless indicated otherwise. An *initial data set* for the Einstein equations is a triple (M, g, k), consisting of a 3-dimensional manifold M (possibly with boundary), a complete Riemannian metric g, along with a symmetric 2-tensor k representing the second fundamental form of an embedding into spacetime. These quantities are assumed to be smooth and satisfy the constraint equations

$$16\pi\mu = R + (\mathrm{Tr}_g k)^2 - |k|^2, \qquad 8\pi J = \mathrm{div}\left(k - (\mathrm{Tr}_g k)g\right), \tag{1.1}$$

where R is the scalar curvature of g, and μ, J are the energy-momentum density of matter fields. The *dominant energy condition* holds if $\mu \geq |J|$. Moreover, the data will be referred to as *asymptotically flat* (AF) if there exists a compact subset $\mathcal{K}$ for which $M \setminus \mathcal{K} = \cup_{a=1}^{a_0} M_{end}^a$, so that the ends M_{end}^a are pairwise disjoint and each is diffeomorphic to the complement of a Euclidean ball $\mathbb{R}^3 \setminus B$. Furthermore, if φ is the diffeomorphism from Euclidean space with Cartesian coordinates x to an end then

$$\begin{aligned} \varphi^* g = \delta + O_2(|x|^{-q}), &\qquad \varphi^* k = O_1(|x|^{-q-1}), \\ \varphi^*\mu, \varphi^* J = O(|x|^{-2q-2}), &\qquad \varphi^* \mathrm{Tr}_g k = O(|x|^{-2q-1}), \end{aligned} \tag{1.2}$$

for some $q > \frac{1}{2}$ where $O_l(|x|^{-q})$ represents a tensor in the weighted space $C^l_{-q}(\mathbb{R}^3)$. Note that the additional decay on the trace of k is usually not included in the AF definition , however it is included here to facilitate the use of [1]. The ADM energy and linear momentum of each end are well-defined [10, 14] with these asymptotics and are given by

$$E = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_i (g_{ij,i} - g_{ii,j})\nu^j dA, \tag{1.3}$$

$$P_i = \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (k_{ij} - (\mathrm{Tr}_g k) g_{ij})\nu^j dA, \tag{1.4}$$

where S_r are coordinate spheres with unit outer normal ν and area element dA. The ADM mass is then the Lorentz length of the energy-momentum vector, that is $m = \sqrt{E^2 - |P|^2}$.

Consider a closed separating hypersurface $\Sigma \subset M$ with null expansions $\theta_\pm = H \pm \mathrm{Tr}_\Sigma k$. Here, H denotes the mean curvature of Σ obtained as the tangential divergence of the unit normal ν pointing towards a designated end M^1_{end}. The null expansions are themselves (spacetime) mean curvatures, namely in the null directions $\nu \pm n$ where n represents the future pointing timelike unit normal to the slice (M, g, k). These quantities can be interpreted physically as determining the rate of change of area for a shell of light emanating from the surface in the outward future/past direction, and thus may be used to assess the strength of the gravitational field. The gravitational field is interpreted as strong near the surface Σ if it is *outer or inner trapped*, that is $\theta_+ < 0$ or $\theta_- < 0$. Moreover, Σ is called a *marginally outer or inner trapped surface* (MOTS or MITS) when $\theta_+ = 0$ or $\theta_- = 0$. These types of surfaces are also referred to as future or past apparent horizons, and naturally arise as boundaries of future or past trapped regions [7]. Moreover, a collection Σ of disjoint MOTS and MITS components will be called an *outermost apparent horizon* with respect to M^1_{end}, if Σ is not enclosed from the perspective of M^1_{end} by any other disjoint collection of apparent horizon components. The existence of an outermost apparent horizon for each end follows from [7], by first finding the outermost MOTS and outermost MITS separately, and then removing components until all are disjoint. Although the outermost MOTS and outermost MITS are individually unique, the second step in which components are removed entails a choice, and thus the resulting outermost apparent horizon may not be unique.

1.3 Definitions of Bartnik type

Definition 1 (b-Admissible Extension). *Let* $(\Omega, \mathrm{g}, \mathrm{k})$ *be an initial data set with nonempty boundary* $\partial\Omega$*, such that* Ω *is either compact or possesses* $a_0 > 0$ *AF ends. Let* **b** *denote a collection of boundary components. A* **b***-admissible extension of* $(\Omega, \mathrm{g}, \mathrm{k})$ *is an AF initial data set* (M, g, k) *without boundary satisfying the following conditions.*

1. (M, g, k) *has one end when* Ω *is compact and has* $a_0 + 1$ *ends otherwise. The additional end will be referred to as the designated end, and will be denoted by* M^1_{end}.

2. *The boundary of the unbounded component of* $M \setminus \Omega$ *consists of the* **b** *boundary components of* Ω.

3. $(\Omega, \mathrm{g}, \mathrm{k})$ *arises from an isometric embedding into* (M, g, k) *which preserves the extrinsic curvature.*

4. (M, g, k) *satisfies the dominant energy condition.*

5. *Any closed minimal hypersurface of* M *is contained within* Ω.

Remark 1. *For technical reasons concerning the proofs of the temporal monotonicity Theorems 3 and 4, it will be convenient to also introduce the following slightly more stringent condition in place of (5).*

5*. *Any closed minimal hypersurface of* M *is contained within the interior int*(Ω) *of* Ω.

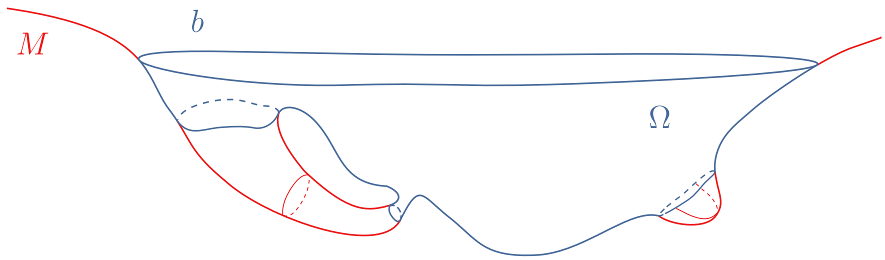

Figure 1: Example of an admissible extension as in Definition 1. The domain Ω is depicted in blue while the extension $M \setminus \Omega$ is shown in red. In this case, the collection of boundary components $\mathbf{b}$ has a single element. Note that the extended manifold (M, g) has all closed minimal surfaces contained in Ω.

In Definition 1, if condition (5) is replaced by (5), we shall refer to the extension as* strictly $\mathbf{b}$-admissible. *Note that while condition (5) admits minimal surfaces as components of $\partial\Omega$, this is no longer the case for (5*).*

Remark 2. *When Ω has several boundary components, the subcollection determined by $\mathbf{b}$ will form the boundary of the noncompact component of $M \setminus \Omega$ containing the designated end, while the remaining components will be 'capped-off' by the compact components of $M \setminus \Omega$. Here 'capping off' includes the possibility that subsets of such remaining boundary components are joined by 'tubes', as long as the resulting construction respects conditions (5) or (5*). See fig. 1 for an example.*

Remark 3. *We avoid defining extensions (M, g, k) with boundaries; the reason is technical and arises from the proof of the temporal monotonicity results Theorems 3 and 4.*

Remark 4. *We recall that in Bartnik's original definition of admissible extensions in the Riemannian setting [11], minimal surfaces are not admitted anywhere (not even inside Ω). A corresponding modification of point 5. of the above definition is viable only if Ω is compact and the extensions have trivial second homology.*

Remark 5. *As to the definition by Huisken and Ilmanen in the Riemannian context [23, Section 9], it relates to the above one with the following differences:*

- *'Strict admissibility' as defined above corresponds to 'admissibility' in [23].*
- *As mentioned in Remark 3 we do not admit extensions with boundaries, in contrast to [23].*

Definition 2 (Bartnik-Type Mass). *Let $(\Omega, \mathrm{g}, \mathrm{k})$ be an initial data set with boundary, such that Ω is either compact or possesses a finite number of AF* *ends. If the set of* **b***-admissible extensions is nonempty, then a Bartnik mass*[1] *of this data is defined to be*

$$m_{\mathbf{b}}(\Omega) = \inf\{m_{ADM}(M^1_{end}, g, k) \mid (M, g, k) \text{ is a } \mathbf{b}\text{-admissible extension of } \Omega\},$$

where $m_{ADM}(M^1_{end}, g, k)$ *is the ADM mass of the designated end.*

We remark that the domain Ω is restricted by the requirement that there exists an admissible extension. (Note that we do not admit infinite values for $m_{\mathbf{b}}$). In particular, this excludes domains with mean concave ($H \leq 0$) boundary. We anticipate that our positivity results Theorems 1 and 2 do not make any further requirement on $\partial\Omega$, while the monotonicity results Theorems 3 and 4 require strict mean convexity ($H \geq 0$).

1.4 Positivity

The positive mass theorem [29, 33] implies that the Bartnik mass is always nonnegative, and the question of its strict positivity outside of the ground state was discussed in Bartnik's original proposal [11, 12]. This has been satisfactorily answered in the time symmetric case by Huisken-Ilmanen [23, Positivity Property 9.1] and Dong-Song [17, Theorem 5.1], see also the discussion in Anderson-Jauregui [2]. However, it does not appear that this question has been previously addressed in the spacetime case. Here we will establish two different strict positivity results, both of which are associated with the presence of an apparent horizon.

Theorem 1 (**Positivity for Compact Data**)**.** *Let* $(\Omega, \mathrm{g}, \mathrm{k})$ *be an initial data set with boundary, such that* Ω *is compact. Assume that a nonempty subset of boundary components* **b** *consists entirely of MOTS and MITS, and that a* **b** *admissible extension exists. Then the corresponding Bartnik mass is strictly positive,* $m_{\mathbf{b}}(\Omega) > 0$*.*

Theorem 2 (**Positivity for Data With AF Ends**)**.** *Let* $(\Omega, \mathrm{g}, \mathrm{k})$ *be an initial data set with boundary, such that* Ω *possesses a finite nonzero number of AF ends. Let* **b** *denote a collection of boundary components. If a* **b** *admissible extension exists, then the corresponding Bartnik mass is strictly positive,* $m_{\mathbf{b}}(\Omega) > 0$*.*

Both theorems may be considered as 'strong field' results in the sense that a MOTS/MITS is always present, either in the boundary of Ω as is the case for Theorem 1, or in any admissible extension of Ω as is the case for Theorem 2 according to the barrier approach for MOTS existence given by [7, Theorem 1.1] and [18, Theorem 1.1].

[1] If **b** consists of all boundary components then the qualifier **b** will be removed from the notation.

1.5 Monotonicity

We now turn our attention to temporal monotonicity properties of the Bartnik mass, a topic which does not appear to have been previously addressed in the literature. We first recall the monotonicity problem for nested domains $\Omega_1 \subset \Omega_2 \subset M$ within a Riemannian manifold M. This holds trivially (in the sense that $m(\Omega_1) \leq m(\Omega_2)$) for Bartnik's original definition [11] and hence for the corresponding modification of Definition 1, cf. Remark 4. However, monotonicity is no longer guaranteed for Definition 1 (or any other definitions which admit minimal surfaces inside Ω.) In this context we recall the monotonicity result of the general (spacetime) Bartnik mass [24], still for nested surfaces within a smooth initial data set. However, this result supposes the existence of stationary minimizing extensions, which are not assumed in the present work. Moreover, such extensions are unlikely to exist for generic, stable MOTS, by analogy with the corresponding result for stable minimal surfaces [26].

In contrast, the next pair of results applies to foliated spacetimes, and requires some preparatory discussion. Consider a 4-dimensional spacetime $\mathcal{M}$ foliated by spacelike hypersurfaces $\{N_t\}_{t\in I}$, where $I \subset \mathbb{R}$ is a nonempty interval. A *marginally outer trapped tube* (MOTT) (adapted to this foliation) is a hypersurface $\mathcal{H} \subset \mathcal{M}$ which is foliated by MOTS $\Sigma_t = \mathcal{H} \cap N_t$. A corresponding *MOTT domain* is a 4-dimensional submanifold of $\mathcal{M}$ with boundary that is foliated by spacelike hypersurfaces $\Omega_t \subset N_t$, such that $\partial\Omega_t = \Sigma_t$.

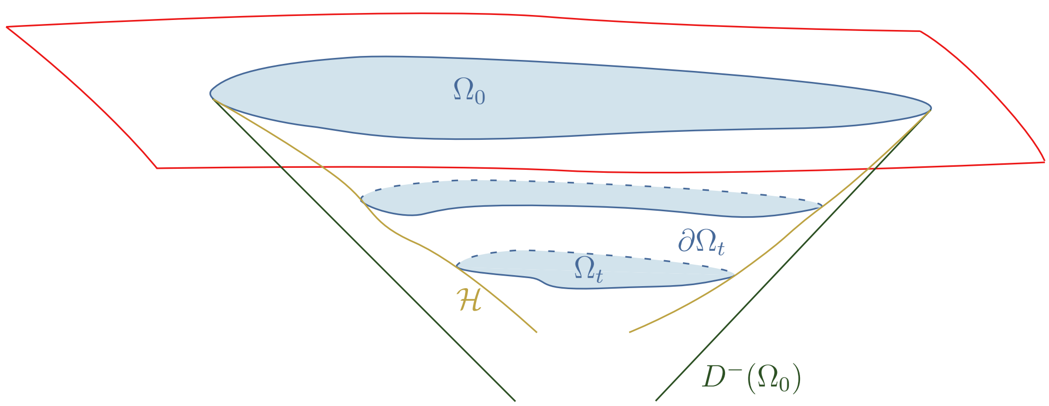

Figure 2: Visualization of the setting of the Monotonicity Theorems 3 and 4. In Theorem 3, the hypersurface $\mathcal{H}$ is the smooth MOTT defined by the past evolution of $\partial\Omega_0$ with respect to the given foliation $(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)_{t\in[-\varepsilon,0]}$. In Theorem 4 $\mathcal{H}$ is any smooth spacelike hypersurface contained in $D^-(\Omega_0)$ and containing $\partial\Omega_0$.

If a spacetime with foliation has the property that some leaf, say N_0, possesses a strictly stable MOTS $\Sigma_0 = \partial\Omega_0$, then the short time existence of a smooth MOTT and corresponding MOTT domain are given by Andersson-

Mars-Simon [5, Theorem 1]. Here, strict stability refers to the property that the principal eigenvalue of the MOTS stability operator [6, Definition 3.1] is strictly positive. If $\mathcal{M}$ satisfies the null energy condition then the MOTT is achronal near Σ_0, and if further the null second fundamental form of Σ_0 with respect to the outward future direction does not vanish identically then the MOTT is spacelike everywhere near Σ_0 [6, Theorem 9.9]. In this situation we will also refer to the associated MOTT domain as spacelike.

Theorem 3 (**Monotonicity Along a MOTT).** *Let $(\Omega_0, \mathrm{g}_0, \mathrm{k}_0)$ be a vacuum initial data set with boundary, such that Ω_0 is compact and each boundary component is a strictly stable and strictly mean convex MOTS. For $\varepsilon > 0$ let $\{(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)\}_{t\in[-\varepsilon,0]}$ be a foliation by compact spacelike hypersurfaces of a spacelike MOTT domain contained within the past domain of dependence $D^-(\Omega_0)$.*

If the initial data has an admissible extension, then for each sufficiently small $t \leq 0$ there exists an admissible extension of $(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)$ and the Bartnik mass is monotonically nondecreasing, that is $m(\Omega_{t_1}) \leq m(\Omega_{t_2})$ for all $t_1 \leq t_2$ sufficiently small.

We note that the condition of strict mean convexity implies that the admissible extensions mentioned above are automatically strictly admissible, by virtue of the maximum principle. This is why we only need to define a single Bartnik-type mass, namely (2), using admissible extensions. It should also be noted that positivity of all Bartnik masses appearing in the monotonicity results is guaranteed by Theorems 1 and 2.

As mentioned in Sect. 1.1 already, Theorem 3 fits the intuitive picture of a 'black hole' which swallows energy upon time evolution. In contrast, the achronal boundary is not specified in the following theorem which, however, is restricted to non-compact data. Of course this theorem holds in particular for MOTTs.

Theorem 4 (**Monotonicity Along an Achronal Tube).** *Let $(\Omega_0, \mathrm{g}_0, \mathrm{k}_0)$ be a vacuum initial data set with strictly mean convex boundary, such that Ω_0 possesses a finite nonzero number of asymptotically flat ends. For $\varepsilon > 0$ let $\{(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)\}_{t\in[-\varepsilon,0]}$ be a foliation by spacelike hypersurfaces of a 4-dimensional submanifold with boundary within the past domain of dependence $D^-(\Omega_0)$, such that each Ω_t contains corresponding asymptotically flat ends and $\{\partial\Omega_t\}_{t\in[-\varepsilon,0]}$ forms a spacelike hypersurface.*

Then the conclusions of Theorem 3 hold: If the initial data has a admissible extension, then for each sufficiently small $t \leq 0$ there exists an admissible extension of $(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)$ and the Bartnik mass is monotonically nondecreasing, that is $m(\Omega_{t_1}) \leq m(\Omega_{t_2})$ for all $t_1 \leq t_2$ sufficiently small.

We note that, compared to the positivity results Theorems 1 and 2, the requirements of Theorems 3 and 4 are more restrictive in three respects: They hold for vacuum only, and $\partial\Omega$ is required to be strictly stable and strictly convex.

Acknowledgements. The authors would like to thank Jan Metzger for discussions at the early stages of this work. MK acknowledges support from NSF Grant DMS-2405045. MM acknowledges support under projects PID2024-158938NB-I00 (MICIU) and SA097P24 (JCyL). The research of WS was funded by the Austrian Science Fund (FWF) [DOI 10.55776/P35078]. For open access purposes, the authors have applied a CC BY public copyright license to any accepted manuscript version arising from this submission.

2 Positivity Proofs

2.1 Proof of Theorem 1

Let (M, g, k) be a **b**-admissible extension (cf Def. 1) of the compact initial data set $(\Omega, \mathrm{g}, \mathrm{k})$ whose **b**-boundary components consist entirely of MOTS and MITS. We recall from Sect. 1.2 the definition and properties of an *outermost apparent horizon* Σ with respect to M^1_{end}. Since **b** is nonempty the outermost apparent horizon is also nonempty, and moreover it separates Ω from infinity. We may then apply the Penrose inequality with suboptimal constant [1, Theorem 1.1] to find

$$m_{ADM}(M^1_{end}, g, k) \geq \sqrt{\frac{\mathcal{A}}{\mathcal{C}}}, \tag{2.1}$$

where $\mathcal{A}$ is the minimum area required to enclose Σ and $\mathcal{C}$ is a positive universal constant. To prove strict positivity of the Bartnik mass, it then suffices to show that $\mathcal{A}$ has a uniform positive lower bound independent of the extension.

To verify the lower bound for $\mathcal{A}$, we will employ a strategy inspired by the proof of [23, Positivity Property 9.1]. Consider a geodesic ball $B_r(p) \subset \Omega$ centered at a point p of radius $r > 0$. Take $r < \mathrm{inj}(p)$ sufficiently small so that $B_{3r}(p)$ is still contained within Ω, and $\partial B_s(p)$ has positive mean curvature for all $s \in (0, 2r]$. By [23, Theorem 1.3 (iii)], there exists a $C^{1,1}$-hypersurface $\mathcal{S} \subset M$ that represents the outermost minimal area enclosure of $\partial B_r(p)$ in M; moreover, this surface is C^∞ and minimal where it does not contact the obstacle $B_r(p)$. By the maximum principle for minimal surfaces, $\mathcal{S}$ cannot lie completely within $B_{2r}(p)$ unless it agrees with $B_r(p)$. Furthermore, the monotonicity formula for minimal surfaces [30, Theorem 17.6] implies that any properly embedded minimal surface within $B_{3r}(p) \setminus B_r(p)$ that intersects $\partial B_{2r}(p)$ must have a uniform area lower bound. The last possibility is that $\mathcal{S}$ lies entirely outside of $B_{2r}(p)$. In this case $\mathcal{S}$ is a closed minimal surface, however since there are no closed minimal surfaces of M that intersect $M \setminus \Omega$, we find that $\mathcal{S} \subset \Omega$. Next extend (Ω, g) smoothly across its boundary to a slightly larger Riemannian manifold (Ω', g') (unrelated to the Bartnik extension) in which

(Ω, g) embeds isometrically; because Ω is compact the geometry of (Ω', g') may be uniformly controlled in terms of that of (Ω, g). In particular, there exists a radius $r_0 > 0$ depending only on (Ω', g'), such that the monotonicity formula can be applied on geodesic balls of radius r_0 centered at points of $\mathcal{S}$, yielding again a uniform area lower bound. Therefore, having considered all possible scenarios for the minimal area enclosure, we find that $|\mathcal{S}| \geq c > 0$ for some constant c depending only on (Ω, g), and since any surface that encloses Σ must also enclose $\partial B_r(p)$ we have $\mathcal{A} \geq |\mathcal{S}| \geq c$.

2.2 Proof of Theorem 2

Let (M, g, k) be a **b**-admissible extension of the initial data set $(\Omega, \mathrm{g}, \mathrm{k})$, and recall that in this setting Ω possesses a finite number of asymptotically flat ends. Since large coordinate spheres in these ends are trapped from the perspective of the designated end M^1_{end}, and large coordinate spheres in the designated end are untrapped, it follows [3, Theorem 3.3] that there exists a MOTS and a MITS homologous to coordinate spheres of M^1_{end}. As discussed above, one may then find an outermost apparent horizon Σ with respect to this designated end. At this stage, the same type of arguments given in the proof of Theorem 1 apply. More precisely, using the Penrose inequality with suboptimal constant it is sufficient to establish uniform positive lower bound for $\mathcal{A}$ independent of the extension, where again $\mathcal{A}$ is the minimal area required to enclose Σ from the perspective of M^1_{end}. This, in turn, may be achieved by estimating the outermost minimal area enclosure $\mathcal{S}$ (with respect to M^1_{end}) of the surface $\mathbf{S}_r$ formed by the union of coordinate spheres having sufficiently large radius r in the asymptotically flat ends of Ω. Observe that $\mathbf{S}_r$ is enclosed by Σ with respect to M^1_{end}, since the maximum principle for MOTS/MITS [7, Proposition 2.4] prevents Σ from entering these ends. Moreover, $\mathbf{S}_r$ has negative mean curvature with respect to the inward pointing normal, and thus $\mathcal{S}$ cannot intersect $\mathbf{S}_r$ and must be a minimal surface. By (5) of Definition 1, it follows that $\mathcal{S}$ lies within the precompact component of $\Omega \setminus \mathbf{S}_r$. Hence, its area may be bounded uniformly from below, yielding a corresponding lower bound for $\mathcal{A}$, as in the proof of Theorem 1.

3 Monotonicity Proofs

3.1 The basic idea

In this section we will establish Theorems 3 and 4. (Cf Fig 2 or a vizualisation of either setting). As these results are restricted to vacuum domains, a natural match would be to restrict to *(strictly) admissible vacuum extensions* M_0 of Ω_0 (defined by replacing the dominant energy condition by vacuum in point 4. of Definiton 1). The basic idea is then to solve the vacuum Einstein

Cauchy problem backward in time, and show that the data induced by the resulting spacetime $D^-(M_0)$ on suitable hypersurfaces M_t extending Ω_t to the designated end yield (strictly) admissible vacuum extensions. In the setting of Theorem 3, the basic result reads as follows.

Proposition 1. *Let $(\Omega_0, \mathrm{g}_0, \mathrm{k}_0)$ be a compact initial data set whose MOTS boundary components are strictly stable, and let (M_0, g_0, k_0) be a strictly admissible vacuum extension. Let $\{(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)\}_{t\in[-\varepsilon,0]}$ be a foliation by compact spacelike hypersurfaces of a spacelike MOTT domain contained within the past domain of dependence $D^-(\Omega_0)$.*
Then there exists a positive $\varepsilon_1 < \varepsilon$ such that each $(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)$ has a strictly admissible vacuum extension (M_t, g_t, k_t) for $t \in [-\varepsilon_1, 0]$.

Proof. Using local existence of the Cauchy problem for the vacuum Einstein equations we can evolve a collar neighborhood $(V, \tilde{g}_0, \tilde{k}_0)$ of $\partial\Omega$ backwards in time to obtain a portion of a vacuum spacetime. Since the MOTT is spacelike, we may then smoothly extend Ω_t into spacetime to obtain an extension $(M_t, \tilde{g}_t, \tilde{k}_t)$, which first passes through the complement of the MOTT domain within $D^-(\Omega_0)$ and then through $D^-(V)$ to eventually coincide with the unbounded component of $(M_0 \setminus V, \tilde{g}_0, \tilde{k}_0)$. This construction of M_t is illustrated schematically in Figure 3. As these extensions share a single asymptotically flat end they have the same ADM mass.

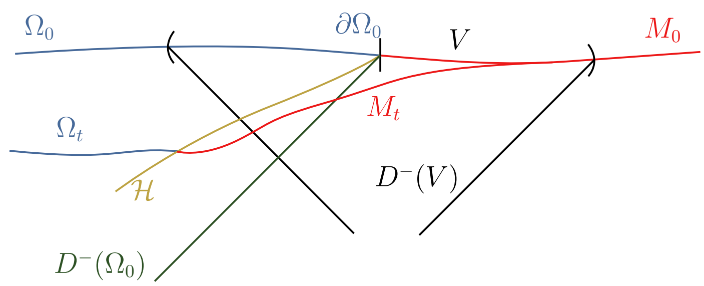

Figure 3: Schematic view of the extension M_t.

It is clear that each M_t satisfies conditions (1)-(4) in definition 1. It remains to verify condition (5*) which will proceed by contradiction. Assume that there is no positive ε_1 such that (5*) holds for all $t \in [-\varepsilon_1, 0]$. There is then a sequence of times $t_i \to 0$, such that (M_{t_i}, g_{t_i}) contains a closed minimal surface that has nontrivial intersection with $M_{t_i} \setminus \mathrm{int}(\Omega_{t_i})$. It follows that the outermost minimal surface $\mathbf{\Sigma}_{t_i} \subset M_{t_i}$, which consists of stable embedded minimal 2-spheres [23, Lemma 4.1 (i)], also has nontrivial intersection with $M_{t_i} \setminus \mathrm{int}(\Omega_{t_i})$. Classical curvature estimates [28, Theorem 3] for stable minimal hypersurfaces, combined with a uniform area bound arising from the outerminimizing property of outermost minimal surfaces as well as the uniformly controlled ambient geometry of M_{t_i}, imply that $\mathbf{\Sigma}_{t_i}$ subconverges smoothly to a closed stable minimal surface $\mathbf{\Sigma}_0 \subset M_0$. Moreover, this limit surface must have nontrivial intersection with $M_0 \setminus \mathrm{int}(\Omega_0)$, yielding a contradiction to strict admissibility of the extension M_0. We conclude that an $\varepsilon_1 > 0$ exists for which $\{(M_t, g_t, k_t)\}_{t\in[-\varepsilon_1,0]}$ are strictly admissible extensions. $\square$

This seems close to the required monotonicity proof in vacuum. However, the value ε_1 is potentially dependent on the chosen extension at time $t = 0$ and might shrink to zero upon minimization over all smooth extensions. This spoils the monotonicity argument. While we were not able to fix this problem within the vacuum class, we succeeded via a substantial detour using 'auxiliary' Einstein-Vlasov matter, which we sketch before giving the proofs.

3.2 Proofs of Theorems 3 and 4

Within the setting of Theorem 3, we first note that this result requires strict mean convexity of $\partial\Omega_0$, which is not required in Proposition 1. In fact the key step 5 of the following proof requires strict mean convexity for all $\partial\Omega_t$, which follows from mean convexity of $\partial\Omega_0$ by shrinking the initial interval $[-\varepsilon, 0)$ if necessary. As a lession from the previous subsection, we avoid any further shrinking of the time interval in the following process (steps 4-6) which consists of modifying the extensions (M_t, g_t, k_t) so that they become strictly admissible for all $t \in [-\varepsilon, 0]$. This comes at the expense of leaving the vacuum class and modeling the given energy-momentum tensor by suitable Einstein-Vlasov particles which we do in step 2. Proposition 1, formulated for vacuum, carries over to Einstein-Vlasov straightforwardly. However, the modeling itself requires a prior deformation of the given data to ones satisfying the strict dominant energy condition (step 1) while preserving strict admissibility. The latter property is shown by a compactness argument analogous to Proposition 1. The key step 5 then requires two further perturbations (steps 3 and 4). All deformations carried out below induce none or only small changes of the ADM mass which in particular do not affect the outcome of the final minimization (step 7).

Proof of Theorem 3. Let (M_0, g_0, k_0) be a strictly admissible extension of the compact initial data set $(\Omega_0, \mathrm{g}_0, \mathrm{k}_0)$ whose boundary components are strictly stable and strictly mean convex MOTS. Let $\{(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)\}_{t\in[-\varepsilon,0]}$ be a foliation by compact spacelike hypersurfaces of a spacelike MOTT domain contained within the past domain of dependence $D^-(\Omega_0)$. By shrinking ε if necessary, we may assume without loss of generality that $\partial\Omega_t$ is strictly mean convex for $t \in [-\varepsilon, 0]$.

1. By Lemma 1 (Appendix A) there exists a perturbation of (g_0, k_0) on $M_0 \setminus \Omega_0$ to strict dominant energy condition, while disturbing the ADM mass only slightly. Denote this new extension by $(M_0, \tilde{g}_0, \tilde{k}_0)$, and note

that for sufficiently small perturbation parameter this is also a strictly admissible extension. Indeed, only the strict admissibility condition (5*) requires some justification, and this is provided by the same type of compactness argument as in Proposition 1.

2. Now apply Lemma 2 (Appendix B) to evolve a collar neighborhood $(V, \tilde{g}_0, \tilde{k}_0)$ of $\partial\Omega$ backwards in time to obtain a portion of a spacetime satisfying the dominant energy condition. Recalling the construction of Proposition 1 (cf Fig. 3), we smoothly extend Ω_t into spacetime to obtain an extension $(M_t, \tilde{g}_t, \tilde{k}_t)$, which first passes through the compliment of the MOTT domain within $D^-(\Omega_0)$ and then through $D^-(V)$ to eventually coincide with the unbounded component of $(M_0 \setminus V, \tilde{g}_0, \tilde{k}_0)$. By abuse of notation going forward, we will denote $(M_t, \tilde{g}_t, \tilde{k}_t)$ by (M_t, g_t, k_t). Note that these extensions (M_t, g_t, k_t) are strictly admissible, but only on the subinterval $[0, \varepsilon_1]$ by Proposition 1.

3. We now perform another perturbation $\delta(t)$ with the goal of preserving the strict dominant energy condition of the data $(M_t, g_{\delta(t)}, k_{\delta(t)})$ on the whole interval $[-\varepsilon, 0]$. We take $\delta(t)$ to be a smooth function on $[-\varepsilon, 0]$ which vanishes at $t = 0$ and is strictly positive for $t < 0$; according to Lemma 1, for each $t < 0$ there exists a perturbation of the unbounded component M_t^∞ of $M_t \setminus \Omega_t$ that yields a new asymptotically flat initial data set $(M_t, g_{\delta(t)}, k_{\delta(t)})$ satisfying: the strict dominant energy condition holds at all points of M_t^∞, and

$$|m_{ADM}(M_t, g_{\delta(t)}, k_{\delta(t)}) - m_{ADM}(M_t, g_t, k_t)| < \delta(t). \qquad (3.1)$$

4. A final perturbation will now turn a certain class of closed, stable minimal surfaces into strictly stable ones. This perturbation concerns U_t, the open subset of M_t^∞ obtained by removing a large coordinate sphere in the asymptotic end and taking the bounded component. By applying White's bumpy metric theorem [32, Theorem 3.1], we can find a nearby smooth metric $g'_{\delta(t)}$ which agrees with $g_{\delta(t)}$ outside of U_t, and has the property that any closed immersed stable minimal surface in $(M_t, g'_{\delta(t)})$ which intersects U_t is strictly stable. Since the deformation carried out in step 4 guarantees that the data $(M_t, g_{\delta(t)}, k_{\delta(t)})$ satisfy the strict dominant energy condition, so do $(M_t, g'_{\delta(t)}, k'_{\delta(t)})$ (where $k'_{\delta(t)} = k_{\delta(t)}$) when the present perturbation is sufficiently is close to $g_{\delta(t)}$. Moreover, the mass is unchanged and therefore (3.1) holds for this perturbation.

5. Recalling Definition 1 of *admissibility* we now claim that the set of times $I \subset [-\varepsilon, 0]$ such that $(M_t, g'_{\delta(t)}, k'_{\delta(t)})$ is an admissible extension of $(\Omega_t, \mathrm{g}_t, \mathrm{k}_t)$, is the entire interval $[-\varepsilon, 0]$. First, observe that conditions (1)-(4) are immediately satisfied, and thus it remains to verify (5). The

case $t=0$ is trivial since $g'_{\delta(0)} = g_0$ and $k'_{\delta(0)} = k_0$, so that I is not empty. Next, notice that the arguments at the beginning of this section show that I is open. More precisely, if this is not the case then there exists $t_* \in I$ and a sequence $t_j \to t_*$ with $t_j \notin I$ for each j. This yields a sequence of stable minimal surfaces, each element of which lies in $(M_{t_j}, g'_{\delta(t_j)})$ and exits Ω_{t_j}. These must then subconverge to a stable minimal surface of $(M_{t_*}, g'_{\delta(t_*)})$, that lies within Ω_{t_*} because $t_* \in I$. On the other hand, since the jth member of the sequence of minimal surfaces intersects $M_{t_j} \setminus \Omega_{t_j}$, the limit minimal surface must have nontrivial intersection with $\partial\Omega_{t_*}$. However, this is impossible by the maximum principle as $\partial\Omega_{t_*}$ is strictly mean convex. To establish closedness, consider a sequence $t_j \in I$ with $t_j \to t_0 \geq -\varepsilon$. If $t_0 \notin I$, then there exists a closed minimal surface of $(M_{t_0}, g'_{\delta(t_0)})$ that exits Ω_{t_0}. It follows that the outermost minimal surface $\mathbf{\Sigma}_{t_0}$ must intersect U_{t_0} as it cannot enter the part of the asymptotic end foliated by mean convex coordinate spheres. Thus, $\mathbf{\Sigma}_{t_0}$ is strictly stable. The implicit function theorem, based at $\mathbf{\Sigma}_{t_0}$, will then yield a closed minimal surface in $(M_{t_j}, g'_{\delta(t_j)})$ for all j sufficiently large, which has nontrivial intersection with $M_{t_j} \setminus \Omega_{t_j}$. This contradicts the assumption that $t_j \in I$, so that in fact $t_0 \in I$ and I is closed, establishing the claim.

6. As final step in the preparation of the data we observe that the admissible extensions $\{(M_t, g'_{\delta(t)}, k'_{\delta(t)})\}_{t\in[-\varepsilon,0]}$ constructed above are in fact strictly admissible. To see this, assume that one of them, M_t, is not strictly admissible. Then there exists a closed minimal surface in M_t that leaves $\mathrm{int}(\Omega_t)$. However, by admissibility this minimal surface must be contained within Ω_t, and hence it has a nontrivial intersection with $\partial\Omega_t$. This contradicts the maximum principle, and establishes the desired property.

7. To address monotonicity of the Bartnik mass, let $t_2 \in [-\varepsilon, 0]$, and choose a function $\delta_2(t)$ that is smooth on $[-\varepsilon, t_2]$ with the property that it vanishes at t_2 and is strictly positive for $t < t_2$. Then applying the above constructions with t_2 playing the role of 'starting time', for any starting strictly admissible extension $(M_{t_2}, g_{t_2}, k_{t_2})$ we obtain strictly admissible extensions $(M_{t_1}, g'_{\delta_2(t_1)}, k'_{\delta_2(t_1)})$ for each $t_1 \in [-\varepsilon, t_2]$ such that

$$|m_{ADM}(M_{t_1}, g'_{\delta_2(t_1)}, k'_{\delta_2(t_1)}) - m_{ADM}(M_{t_2}, g_{t_2}, k_{t_2})| < \delta_2(t_2). \tag{3.2}$$

Thus, we have shown that for any $t_1 \in [-\varepsilon, t_2]$ there exists a strictly admissible extension M_{t_1} having an asymptotic end whose ADM mass is arbitrarily close to that of M_{t_2}. It follows immediately that the Bartnik masses satisfy $m(\Omega_{t_1}) \leq m(\Omega_{t_2})$.

□

Proof of Theorem 4. Theorem 4 can now be established in a similar manner with straightforward modifications. □

Remark 6. *This remark expands the discussion at the beginning of Sect. 1.5 where we emphasized the difference between temporal monotonicity on the one hand, and monotonicity of nested domains within a $t = const = 0$ slice M_0 on the other hand. However, these issues are related when we consider the limit that $\mathcal{H}$ approaches M_0 in the former setting. Then in particular Theorem 4 continues to hold and guarantees monotonicity of nested domains within Ω_0 having boundaries sufficiently close to $\partial\Omega_0$, as the requirement of strict mean convexity of $\partial\Omega_0$ excludes adjacent minimal surfaces.*

Remark 7. *This final comment concerns a possible extension of Theorem 3 to the case where the outermost MOTT $\mathcal{H} = \{\partial\Omega\}$ is not smooth anymore. In a spacetime foliated by spacelike slices where some initial slice M_0 contains a MOTS, the future propagation will generically reveal 'jumps' of the outermost MOTT. A systematic investigation of this behavior has been carried out in [4]. Regarding monotonicity of $m(\Omega)$ along such a jump, it holds provided minimal surfaces are absent on M_t (since any admissible extension of the target MOTS of the jump yields an admissible extension of the origin). In the generic case, however, we expect the monotonicity problem to be very intricate.*

A Appendix

In the proof of Theorems 3 and 4, a technical perturbation to strict dominant energy condition is utilized, in which the ADM mass is also only slightly disturbed. It follows from a combination of existing perturbation results, and is recorded here for completeness.

Lemma 1. *Let (M, g, k) be an asymptotically flat initial data set with one end and compact boundary, satisfying the dominant energy condition. For any $\delta > 0$, there exists a new asymptotically flat initial data set $(M, \tilde{g}, \tilde{k})$ admitting the following properties.*

1. *$(\tilde{g}, \tilde{k})$ is δ-close to (g, k) in weighted Hölder space involving at least two derivatives. In particular, $(\tilde{g}, \tilde{k})$ agrees with (g, k) up to second order at the boundary of M.*

2. *The strict dominant energy condition holds globally for $(\mathrm{int}(M), \tilde{g}, \tilde{k})$.*

3. *The relation between the two ADM masses is given by*

$$|m_{ADM}(M, \tilde{g}, \tilde{k}) - m_{ADM}(M, g, k)| < \delta. \tag{A.1}$$

Proof. Choose a large coordinate sphere S_r in the asymptotic end, and denote the bounded component of $M \setminus S_r$ by M_r. By [16, Theorem 1.4] there exists a nearby new asymptotically flat initial data set (M, g_r, k_r) which agrees with (M, g, k) on M_r, and agrees with Kerr-Newman initial data on $M \setminus M_{2r}$. Moreover, the mass of the new data may be made as close to the original as desired by taking r sufficiently large. Choosing the Kerr-Newman data to have nonvanishing electromagnetic fields then yields a strictly dominant energy condition for the new data on $M \setminus M_{2r}$. Next, we deform (M, g_r, k_r) by using a slight generalization of the statement in [22, Theorem 8], to obtain the desired data set $(M, \tilde{g}, \tilde{k})$. In particular, $(\tilde{g}, \tilde{k})$ agree with (g_r, k_r) on $M \setminus M_{3r}$ and admit a strict dominant energy condition on M_{3r}. This is achieved by applying [22, Theorem 8] on the compact closure $\bar{M}_{3r}$ and choosing a small ball $B \subset M_{3r} \setminus M_{2r}$ to obtain: given $\varepsilon > 0$ we may choose a sufficiently small positive function $u \in C^0_c(\mathrm{int}(M_{3r}))$ such that

$$\tilde{\mu} - |\tilde{J}| \geq (\mu_r - |J_r|) + u - \varepsilon \mathbf{1}_B \qquad \text{on } M_{3r}, \tag{A.2}$$

where $\mathbf{1}_B$ is the indicator function for B, and $(\tilde{\mu}, \tilde{J})$, (μ_r, J_r) are the energy-momentum density of matter fields for $(M, \tilde{g}, \tilde{k})$, (M, g_r, k_r) respectively. Since $\mu_r - |J_r| \geq c > 0$ for some constant c on $M_{3r} \setminus M_{2r}$, by taking ε small enough it follows that $\tilde{\mu} - |\tilde{J}| > 0$ on M. We note that the published statement of [22, Theorem 8] allows for a positive u only on any $V \Subset M_{3r}$. However, this is based on [22, Theorem 3.3], whose proof in turn is essentially the same as [16, Theorem 3.1]. This latter result is stated using certain weighted Hölder space instead of compactly contained domains. Therefore, a slight generalization of [22, Theorem 8] utilizing the weighted Hölder spaces is also valid, and this is what we have employed here. In fact, Huang-Lee make reference to such a generalization for [22, Theorem 3.3] in the first paragraph of its proof. □

B Appendix

In this section we will show how to realize certain types of initial data in the context of Vlasov matter, with the purpose of embedding portions of initial data satisfying the dominant energy condition into a Lorentzian manifold satisfying the spacetime dominant energy condition. It should be noted that Glöckle [19, Theorem 4] has shown that there exist smooth DEC initial data sets which do not arise as the induced metric and second fundamental from for a spacelike slice within a smooth spacetime satisfying the *spacetime dominant energy condition*, namely that $T(X, Y) \geq 0$ for any two future causal vectors X, Y where T is the stress-energy tensor. We begin with two preliminary propositions.

Proposition 2 (Strict DEC realization by smooth Vlasov data). *Let (M, g, k) be an initial data set satisfying the strict dominant energy condition $\mu > |J|$*

on a compact subset $\mathcal{K} \subset M$. Then there exists a nonnegative (Vlasov distribution) function $f_0 \in C^\infty(TM|_{\mathcal{K}})$ such that

1. *$f_0(x,\cdot)$ is compactly supported in each fiber $T_x\mathcal{K}$;*

2. *for every $x \in \mathcal{K}$ the following identities hold*

$$\int_{T_xM} f_0(x,v)\,\sqrt{1+|v|_g^2}\,dv_{g_x} = \mu(x), \qquad \int_{T_xM} f_0(x,v)\,v\,dv_{g_x} = J(x). \quad \text{(B.1)}$$

Proof. Since $\mathcal{K}$ is compact we find $\sigma := \min_{\mathcal{K}}(\mu - |J|) > 0$. Choose a nonnegative smooth function $\chi \in C_c^\infty([0,\infty))$, supported in $[0, r_0^2]$ for some $r_0 > 0$, and normalize it so that if $\eta(x,v) := \chi(|v|_g^2)$ then

$$\int_{T_xM} \eta(x,v)\,dv_{g_x} = 1 \qquad \text{for all } x \in \mathcal{K}. \quad \text{(B.2)}$$

Because $\eta(x,\cdot)$ is radial in each fiber, one also has

$$\int_{T_xM} \eta(x,v)\,v\,dv_{g_x} = 0 \qquad \text{for all } x \in \mathcal{K}. \quad \text{(B.3)}$$

Next, for $q \in T_xM$, define the translated bump $\eta_q(x,v) := \eta(x, v-q)$ and observe that

$$\int_{T_xM} \eta_q(x,v)\,dv_{g_x} = 1, \qquad \int_{T_xM} \eta_q(x,v)\,v\,dv_{g_x} = q. \quad \text{(B.4)}$$

Since $\eta_q(x,\cdot)$ is supported in the g_x-ball $B_{r_0}(q) \subset T_xM$, we have

$$\sqrt{1+|v|_g^2} \le |q|_g + \sqrt{1+R^2} \qquad \text{on } \operatorname{supp}\eta_q(x,\cdot), \quad \text{(B.5)}$$

and therefore

$$\mathcal{E}(x,q) := \int_{T_xM} \eta_q(x,v)\,\sqrt{1+|v|_g^2}\,dv_{g_x} \le |q|_g + \sqrt{1+r_0^2}. \quad \text{(B.6)}$$

By choosing $c > 0$ so large that $c^{-1}\sqrt{1+r_0^2} < \sigma$, it follows that for every $x \in \mathcal{K}$ we have

$$\frac{1}{c}\,\mathcal{E}(x, cJ(x)) \le |J(x)| + c^{-1}\sqrt{1+r_0^2} < |J(x)| + \sigma \le \mu(x). \quad \text{(B.7)}$$

Hence the function

$$a(x) := \frac{\mu(x) - c^{-1}\mathcal{E}(x, cJ(x))}{e_0}, \qquad e_0 := \int_{T_xM} \eta(x,v)\,\sqrt{1+|v|_g^2}\,dv_{g_x}, \quad \text{(B.8)}$$

is smooth and nonnegative on $\mathcal{K}$; note that e_0 is independent of x. We may now set

$$f_0(x,v) := a(x)\,\eta(x,v) + \frac{1}{c}\,\eta_{cJ(x)}(x,v). \tag{B.9}$$

Then $f_0 \geq 0$, and for each $x \in \mathcal{K}$ the function $f_0(x,\cdot)$ is compactly supported in T_xM. To compute the moments, observe that

$$\int_{T_xM} f_0(x,v)\,v\,dv_{g_x} = a(x)\int_{T_xM}\eta(x,v)\,v\,dv_{g_x} + \frac{1}{c}\int_{T_xM}\eta_{cJ(x)}(x,v)\,v\,dv_{g_x}. \tag{B.10}$$

so that according to the identities above

$$\int_{T_xM} f_0(x,v)\,v\,dv_{g_x} = 0 + \frac{1}{c}\,(cJ(x)) = J(x). \tag{B.11}$$

Moreover

$$\int_{T_xM} f_0(x,v)\,\sqrt{1+|v|_g^2}\,dv_{g_x} = a(x)e_0 + \frac{1}{c}\,\mathcal{E}(x,cJ(x)) = \mu(x). \tag{B.12}$$

Thus f_0 is a smooth nonnegative Vlasov datum on $TM|_{\mathcal{K}}$, with the prescribed energy and momentum moments. □

Proposition 3 (Vlasov realization up to a flat vacuum boundary)**.** *Let (M,g,k) be an initial data set, and let $\mathcal{K} \subset M$ be a compact domain with smooth boundary. Assume that $\mu > |J|$ on $\mathrm{int}(\mathcal{K})$, and that both μ and J vanish to infinite order along a collection of boundary components $\partial'\mathcal{K} \subset \partial\mathcal{K}$. Then there exists a nonnegative function $f_0 \in C^\infty\big(TM|_{\mathcal{K}}\big)$ such that*

1. *f_0 is compactly supported in each fiber T_xM;*
2. *f_0 vanishes to infinite order along $T(\partial'\mathcal{K})$, hence extends by zero to a smooth nonnegative function on TM;*
3. *for every $x \in \mathcal{K}$ the following identities hold*

$$\int_{T_xM} f_0(x,v)\,\sqrt{1+|v|_g^2}\,dv_{g_x} = \mu(x), \qquad \int_{T_xM} f_0(x,v)\,v\,dv_{g_x} = J(x). \tag{B.13}$$

Proof. Choose a smooth boundary defining function

$$s \in C^\infty(\mathcal{K}), \qquad s \geq 0, \qquad \partial'\mathcal{K} = \{s = 0\}, \qquad \mathrm{int}(\mathcal{K}) = \{s > 0\}. \tag{B.14}$$

Let $\{\chi_n\}_{n\geq 1}$ be a dyadic partition of unity so that

$$\begin{aligned} \sum_{n=1}^{\infty}\chi_n &= 1 \quad \text{on } \mathrm{int}(\mathcal{K}), \\ \operatorname{supp}\chi_n &\subset \{2^{-n-2-n_0} < s < 2^{-n-n_0}\} \quad \text{for } n \geq 2, \end{aligned} \tag{B.15}$$

and with χ_1 supported away from $\partial'\mathcal{K}$; here n_0 is fixed sufficiently large so that the annular regions are smooth and contained within $\mathcal{K}$. For each $n \geq 0$ let $\mathcal{K}_n \Subset \operatorname{int}(\mathcal{K})$ be the closure of the annular region appearing in (B.15), so that $\operatorname{supp} \chi_n \subset \mathcal{K}_n$. Since $\mu > |J|$ on $\operatorname{int}(\mathcal{K})$, for each n there is a positive constant $\sigma_n := \min_{\mathcal{K}_n}(\mu - |J|) > 0$. Hence, on each fixed compact set $\mathcal{K}_n$, the strict dominant energy condition holds with a uniform positive gap. By the strict-DEC realization Proposition 2, there exist nonnegative smooth functions $h_n \in C^\infty(TM|_{\mathcal{K}_n})$ which are compactly supported in each fiber and satisfy

$$\int_{T_xM} h_n(x,v)\sqrt{1+|v|_g^2}\,dv_{g_x} = \mu(x), \qquad \int_{T_xM} h_n(x,v)\,v\,dv_{g_x} = J(x), \tag{B.16}$$

for all $x \in \mathcal{K}_n$.

Let $\pi : TM \to M$ be the bundle projection, and set $f_n := (\chi_n \circ \pi)\, h_n$. Then $f_n \geq 0$, f_n is smooth on $TM|_{\mathcal{K}_n}$, and by linearity of the moment map

$$\int_{T_xM} f_n(x,v)\sqrt{1+|v|_g^2}\,dv_{g_x} = \chi_n(x)\mu(x), \tag{B.17}$$

$$\int_{T_xM} f_n(x,v)\,v\,dv_{g_x} = \chi_n(x)J(x). \tag{B.18}$$

Now define

$$f_0 := \sum_{n=1}^{\infty} f_n \qquad \text{on } TM|_{\operatorname{int}(\mathcal{K})}. \tag{B.19}$$

Because the partition is locally finite on $\operatorname{int}(\mathcal{K})$, this is pointwise well-defined and smooth in the interior. Furthermore, its moments are given by

$$\sum_{n=1}^{\infty} \chi_n\mu = \mu, \qquad \sum_{n=1}^{\infty} \chi_n J = J, \tag{B.20}$$

hence f_0 realizes exactly the prescribed pair (μ, J) on $\operatorname{int}(\mathcal{K})$.

It remains to establish smooth extension by zero across $\partial'\mathcal{K}$. Since μ and J vanish to infinite order along $\partial'\mathcal{K}$, for every $N \geq 1$ and every integer $l \geq 0$ there is a constant $C_{N,l}$ such that on the collar $\{2^{-n-2-n_0} < s < 2^{-n-n_0}\}$ we have

$$\|\mu\|_{C^l} + \|J\|_{C^l} \leq C_{N,l}\, 2^{-nN}. \tag{B.21}$$

The strict-DEC realization on each fixed compact annulus depends smoothly on the data, so that the corresponding h_n satisfy analogous tame estimates

$$\|h_n\|_{C^l} \leq C'_{N,l}\, 2^{-nN}. \tag{B.22}$$

Since the derivatives of χ_n grow at most polynomially in 2^n, the series $\sum_{n=1}^{\infty} f_n$ converges in every C^l-norm up to the boundary $T(\partial'\mathcal{K})$, and the limit vanishes to infinite order there. □

We will now evolve the Einstein-Vlasov system near a vacuum/nonvacuum interface, in order to realize certain types of DEC initial data as spacelike hypersurfaces in DEC spacetimes. Although Glöckle [19, Theorem 4] has found examples where this is not possible, they require conditions that are far from the setting of our next result. In particular, the counterexamples need an open subset on which $\mu = |J|$ together with a lack of C^2 regularity for the map

$$x \mapsto \begin{cases} \mu^{-1}(x)(J(x) \otimes J(x)) & \mu(x) \neq 0, \\ 0 & \mu(x) = 0. \end{cases} \tag{B.23}$$

Lemma 2. *Let (M, g, k) be a bounded initial data set, and let $\Omega \subset M$ be a compact domain with smooth boundary. Assume that the data are vacuum ($\mu = |J| = 0$) on Ω, and satisfy the strict dominant energy condition $\mu > |J|$ on $M \setminus \Omega$. Then there exists a short time smooth Einstein-Vlasov development realizing the given initial data on M as a spacelike hypersurface, and satisfying the spacetime dominant energy condition.*

Proof. Consider the compact domain $\mathcal{K} := \overline{M} \setminus \mathrm{Int}(\Omega)$. Since the strict dominant energy condition holds on $\mathrm{int}(\mathcal{K}) = M \setminus \Omega$, and μ and J vanish to infinite order on $\partial\Omega =: \partial'\mathcal{K}$, we may apply Proposition 3 to find a smooth nonnegative function $\widetilde{f_0} \in C^\infty\big(TM|_{\mathcal{K}}\big)$ which is compactly supported in each fiber, vanishes to infinite order along $T(\partial\Omega)$, and satisfies $\mu_{\widetilde{f_0}} = \mu$ and $J_{\widetilde{f_0}} = J$ on $\mathcal{K}$. Next, extend $\widetilde{f_0}$ by zero over $T\Omega$ to obtain a smooth nonnegative function $f_0 \in C^\infty(TM)$ such that $f_0 = 0$ on $T\Omega$. Moreover, on Ω one has $\mu_{f_0} = \mu = 0$ and $J_{f_0} = J = 0$. Hence, the energy-momentum densities arising from the Vlasov distribution function, and initial data, agree on all of M. We then have that the triple (M, g, k, f_0) satisfies the Einstein-Vlasov constraint equations, and by the standard local Cauchy theory for the Einstein-Vlasov system, there exists a short time smooth spacetime development [8, Theorem 1] realizing the given initial data on M as a spacelike hypersurface. Moreover, since $f_0 \geq 0$ the resulting spacetime satisfies the dominant energy condition. □

References

[1] B. Allen, E. Bryden, D. Kazaras, and M. Khuri, *Proof of the Penrose conjecture with suboptimal constant*, preprint, 2025. arXiv 2504.10641

[2] M. Anderson, and J. Jauregui, *Embeddings, immersions and the Bartnik quasi-local mass conjectures*, Ann. Henri Poincaré, **20** (2019), 1651–1698.

[3] L. Andersson, M. Eichmair, and J. Metzger, *Jang's equation and its applications to marginally trapped surfaces*, in *Complex analysis and dynamical systems IV. Part 2*, 13–45, Contemp. Math. Israel Math. Conf. Proc., **554**, Amer. Math. Soc., Providence, RI.

[4] L. Andersson, M. Mars, J. Metzger, and W. Simon, *The time evolution of marginally trapped surfaces*, Classical Quantum Gravity, **26** (2009), no. 8, 085018, 14 pp.

[5] L. Andersson, M. Mars, and W. Simon, *Local existence of dynamical and trapping horizons*, Phys. Rev. Lett., **95** (2005), 111102.

[6] L. Andersson, M. Mars, and W. Simon, *Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes*, Adv. Theor. Math. Phys., **12** (2008), no. 4, 853–888.

[7] L. Andersson, and J. Metzger, *The area of horizons and the trapped region*, Comm. Math. Phys., **290** (2009), no. 3, 941–972.

[8] H. Andréasson, *The Einstein-Vlasov system/kinetic theory*, Living Rev. Relativ., **5** (2002), 2002-7, 33 pp.

[9] A. Ashtekar, and B. Krishnan, *Dynamical horizons and their properties*, Phys. Rev. D, **68** (2003), 104030.

[10] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math., **39** (1986), no. 5, 661–693.

[11] R. Bartnik, *New definition of quasilocal mass*, Phys. Rev. Lett., **62** (1989), 2346.

[12] R. Bartnik, *Energy in general relativity*, Tsing Hua lectures on geometry and analysis, (Hsinchu 1990-91), 5–27, International Press, Cambridge, MA 1997.

[13] R. Bartnik, *Mass and 3-metrics of non-negative scalar curvature*, Proceedings of the International Congress of Mathematicians, Beijing, **II** (2002), 231–240, Higher Ed. Press.

[14] P. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological Properties and Global Structure of Space-Time (P. Bergmann and V. de Sabbata, eds.), Plenum Press, New York, 1986, pp. 49–59.

[15] P. T. Chruściel, E. Delay, G. J. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Ann. Henri Poincaré **2** (2001), no. 1, 109—178

[16] J. Corvino, and L.-H. Huang, *Localized deformation for initial data sets with the dominant energy condition*, Calc. Var. Partial Differential Equations, **59** (2020), no. 1, Paper No. 42, 43 pp.

[17] C. Dong, and A. Song, *Stability of Euclidean 3-space for the positive mass theorem*, Invent. Math., **239** (2025), no. 1, 287–319.

[18] M. Eichmair, *The Plateau problem for marginally trapped surfaces*, J. Differential Geom., **83** (2009), no. 3, 551–584.

[19] J. Glöckle, *Initial data sets with dominant energy condition admitting no smooth DEC spacetime extension*, Proc. Amer. Math. Soc. Ser., B **11** (2024), 481–488.

[20] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time,* Cambridge University Press, Cambridge, 1973.

[21] S. Hayward, *General laws of black-hole dynamics*, Phys. Rev. D, **49** (1994), 6467.

[22] L.-H. Huang, and D. Lee, *Bartnik mass minimizing initial data sets and improvability of the dominant energy scalar*, J. Differential Geom., **126** (2024), no. 2, 741–800.

[23] G. Huisken, and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., **59** (2001), no. 3, 353–437.

[24] S. McCormick, and P. Miao, *On the evolution of the spacetime Bartnik mass*, Pure and Applied Mathematics Quarterly **15**, (2020) no. 3, 897–920

[25] S. McCormick, *An overview of Bartnik's quasi-local mass*, Beijing J. of Pure and Appl. Math. **1**, (2024) no. 2, 455–487.

[26] C. Mantoulidis, and R. Schoen, *On the Bartnik mass of apparent horizons* Journal of Classical and Quantum Gravity **32**, (2015), no. 20, 205002

[27] R. Penrose, *Some unsolved problems in classical general relativity*, Ann. of Math. Stud., No. 102, Princeton University Press, Princeton, NJ, 1982, pp. 631–668.

[28] R. Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds*, Ann. of Math. Stud., **103**, Princeton University Press, Princeton, NJ, 1983, 111–126.

[29] R. Schoen, and S.-T. Yau, *Proof of the positive mass theorem II*, Comm. Math. Phys., **79** (1981), 231–260.

[30] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ., **3**, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[31] L. Szabados, *Quasi-local energy-momentum and angular momentum in general relativity*, Living Rev. Rel, **12** (2009), no. 4.

[32] B. White, *On the bumpy metrics theorem for minimal submanifolds*, Amer. J. Math., **139** (2017), no. 4, 1149–1155.

[33] E. Witten, *A simple proof of the positive energy theorem*, Comm. Math. Phys., **80** (1981), no. 3, 381–402.