Roundness of the ample cone and existence of double Lagrangian fibrations on hyperkähler manifolds

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Abstract. Let M be a hyperkähler manifold of maximal holonomy (that is, an IHS manifold), and let K be its Kähler cone, which is an open, convex subset in the space $H^{1,1}(M,\mathbb{R})$ of real (1,1)-forms. This space is equipped with a canonical bilinear symmetric form of signature $(1,n)$ obtained as a restriction of the Bogomolov-Beauville-Fujiki form. The set of vectors of positive square in the space of signature $(1,n)$ is a disconnected union of two convex cones. The "positive cone" is the component which contains the Kähler cone. We say that the Kähler cone is "round" if it is equal to the positive cone. The manifolds with round Kähler cones have unique bimeromorphic model and correspond to Hausdorff points in the corresponding Teichmüller space. We prove thay any maximal holonomy hyperkähler manifold with $b_2 > 5$ has a deformation with round Kähler cone and the Picard lattice is of signature (1,1), admitting two non-collinear integer isotropic classes. This is used to show that all known examples of hyperkähler manifolds admit a deformation with two transversal Lagrangian fibrations, and their Kobayashi metric vanishes.

1 Introduction

This paper gives a simple solution for a construction problem of hyperkähler geometry. We construct a hyperkähler manifold with rank 2 hyperbolic Picard lattice and maximal possible Kähler cone.

For our purposes, "a hyperkähler manifold" is a compact, holomorphic symplectic compact manifold M of Kähler type, which satisfies "the maximal holonomy condition", that is, $\pi_1(M) = 0$, $\dim H^{2,0}(M) = 1$. This condition is also known as IHS ("irreducible holomorphic symplectic").

The shape of the Kähler cone of a hyperkähler manifold is more or less understood by now (see [AV3]). However, finding examples of manifolds with prescribed shape of their Kähler cone is a complicated task. Such constructions are either very explicit or based on convoluted arguments from number theory. The automorphism group of a hyperkähler manifold and its set of Lagrangian fibrations can be described explicitly in terms of its periods and the shape of the Kähler cone ([AV4]). Therefore, finding manifolds with prescribed Kähler cones has many practical applications.

Recall that the second cohomology of a maximal holonomy hyperkähler manifold is equipped with a bilinear symmetric form of signature $(3, b_2-3)$, which is

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essentially of topological origin ([Bea, Bo, F]). This form, denoted by q further on, is called **the Bogomolov-Beauville-Fujiki form**; it is positive on the Kähler cone, and has signature $(1, b_2 - 3)$ on $H^{1,1}(M, \mathbb{R})$.

This implies that the set of "positive vectors" (the vectors with positive square) in $H^{1,1}(M, \mathbb{R})$ has two connected components, both of them convex cones. However, only one of these two components may contain Kähler forms. We call this component "the positive cone" of a hyperkähler manifold.

Every hyperkähler manifold is equipped with a collection $\mathfrak{S}$ of primitive integer cohomology classes in $H^2(M, \mathbb{Z})$ with negative squares, called **the MBM classes** ([AV1]). This set is invariant on deformations of M and under the action of the monodromy group Γ generated by the monodromy of the Gauss-Manin connection for all deformations of M. The group Γ (originally defined by E. Markman, [Mar1], who called it **the monodromy group**) is mapped to the orthogonal lattive $O(H^2(M, \mathbb{Z}))$ with finite kernel, and its image is a finite index sublattice in $O(H^2(M, \mathbb{Z}))$ ([V1]).

In [AV2], it was shown that the monodromy group Γ acts on the set of MBM classes with a finite number of orbits, which were computed for some deformational classes of hyperkähler manifolds in [BM], [HT3], [HT4], [AV5].

As shown in [AV3] (the result is essentially due to E. Markman, [Mar2]), the positive cone $\operatorname{Pos}(M, I)$ of a hyperkähler manifold is cut into pieces by hyperplanes orthogonal to the MBM classes which lie in $H^{1,1}(M, I)$, and each of the connected components of this complement can be realized as a Kähler cone of a certain hyperkähler birational model of (M, I). In other words, the Kähler cone is a connected component of the set

$$\operatorname{Pos}^\circ(M, I) := \operatorname{Pos}(M, I) \Bigg\backslash \bigcup_{\alpha \in \mathfrak{S} \cap H^{1,1}(M,I)} \alpha^\perp , \qquad (1.1)$$

where $\mathfrak{S}$ is the set of all MBM classes in $H^2(M, \mathbb{Z})$, and all connected components are realized as Kähler cones for birational models of (M, I)

The authomorphism group of a hyperkähler manifold (M, I) is expressed in terms of its Kähler cone and the monodromy group as follows. Let $\Gamma_I \subset \Gamma$ be the subgroup of the monodromy group preserving the Hodge decomposition on $H^2(M)$. Then $\operatorname{Aut}(M, I)$ is a subgroup of all elements in Γ_I which preserve the Kähler cone.

We say that a manifold M **has round Kähler cone** if $\operatorname{Kah}(M) = \operatorname{Pos}(M)$, or, equivalently, when the set of MBM classes in $H^{1,1}(M, I)$ is empty.

To construct a manifold with round Kähler cone and rank 2 Picard lattice, we use Kneser's orbit theorem, claiming that for any non-degenerate quadratic lattice Λ, the orthogonal group $O(\Lambda)$ acts on the set of non-degenerate sublattices with given discriminant and rank with finitely many orbits (Theorem 3.1).

A rank 2 integer quadratic lattice is called **hyperbolic** if it is generated by 2 isotropic vectors,[1] in other words, if its intersection lattice in an appropriate

[1] A vector $x \in \Lambda$ in a lattice (Λ, q) is **isotropic** if $q(x, x) = 0$.

basis has the form $\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$. Such a hyperbolic lattice is denoted by $U(N)$.

An **MBM bound** for a hyperkähler manifold is the number

$$C := \max\{-q(x,x) \ \mid \ x \in \mathfrak{S}\},$$

where $\mathfrak{S}$ denotes the set of MBM classes. Since $\Gamma \subset O(H^2(M,\mathbb{Z}))$ acts on $\mathfrak{S}$ with finitely many orbits (by [AV2]), this number is finite.

Clearly, for all vectors x in $U(N)$, the square (x,x) is divisible by N. Therefore, any primitive sublattice $U(N) \subset H^2(M,\mathbb{Z})$ with $N > C$ contains no MBM classes and has round Kähler cone. However, if $H^2(M,\mathbb{Z})$ does not contain primitive lattices isomorphic to $U(N)$ for all $N > C$, then the set of $O(H^2(M,\mathbb{Z}))$-classes of primitive hyperbolic sublattices in $H^2(M,\mathbb{Z})$ is finite, by Kneser's theorem. This easily leads to a contradiction (Theorem 3.2).

2 Kobayashi metric on hyperkähler manifolds

We apply the results of the current paper to the vanishing of the Kobayashi pseudometric on hyperkähler manifolds. In this section we summarize some of our results in [KLV] joint with S. Lu. The aim of the current paper is to imrove some of the bounds imposed on the Betti numbers, and also to show vanishing of the Kobayashi pseudometric for all of the known compact hyperkähler examples.

Definition 2.1: An **ergodic complex structure** is a complex structure I on M such that for any complex structure I' in the same deformation class there exists a sequence of diffeomorphisms $\nu_i \in \operatorname{Diff}(M)$ such that $\lim_i \nu_i(I) = I'$, where the limit of $\nu_i(I) \in \operatorname{End}(TM)$ is taken with respect to the C^∞-topology on the space of tensors. We denote the space of all integrable complex structures with this topology by Comp.

Theorem 2.2: Any complex structure of hyperkähler type on a hyperkähler manifold with $b_2 \geqslant 5$ with $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M,\mathbb{Q}) = 0$ is ergodic.
Proof: [V2, V2bis]). ■

We will need another diffeomorphism orbit, which is smaller than the maximal one, but has many of the same properties.

Theorem 2.3: Let (M,I) be a hyperkähler manifold such that $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M,\mathbb{Q})$ is a rank 1 space generated by a class $\alpha \in H^2(M,\mathbb{Q})$, and $\operatorname{Teich}_\alpha$ the Teichmüller space of all complex structure with $\alpha \in (H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M,\mathbb{Q})$ and deformationally equivalent to I. Then $\operatorname{Diff}(M) \cdot I$ is dense in $\operatorname{Teich}_\alpha$.
Proof: [V2bis, Theorem 2.5, Theorem 3.1]. ■

Theorem 2.4: [KLV] Let (M,I) be a complex manifold with vanishing Kobayashi pseudometric. Then the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class. Moreover, for each complex structure I_1 such that the closure of $\operatorname{Diff}(M)I_1$ in Comp contains I, the pseudometric on (M, I_1) also vanishes.

Proof: The proof follows easily from semicontinuity of the diameter of the Kobayashi pseudometric, considered as a function on Comp ([KLV]). ■

Theorem 2.5: [KLV] Let M be a hyperkähler manifold admitting two Lagrangian fibrations associated with two non-proportional parabolic classes. Then the Kobayashi pseudometric on M vanishes.

To prove that a given hyperkähler manifold admits a deformation with two distinct Lagrangian fibrations, in [KLV] we used an argument based on [AV4].

Theorem 2.6: [KLV] Let M be a maximal holonomy hyperkähler manifold with $b_2(M) > 13$. Then M admits a projective deformation with Picard lattice of signature $(1, 2)$, with round Kähler cone (that is, with the Kähler cone equal to the positive cone), and its automorphism group has finite index in the arithmetic group $SO(\operatorname{Pic}(M))$ of orthogonal automorphisms of its Picard lattice.

Proof: From [AV4, Theorem 3.11] it follows that there exists a projective deformation with Picard rank 3, isotropic classes in $H^{1,1}(M) \cap H^2(M, \mathbb{Q})$ and without MBM classes of type (1,1). From [AV4, Theorem 2.10] it follows that for such a manifold the Kähler cone is equal to the positive cone, and from [AV4, Theorem 2.6, Theorem 2.7, Corollary 2.12] it follows that its automorphism group has finite index in the arithmetic group $SO(\operatorname{Pic}(M))$. ■

Theorem 2.7: [KLV] Let M be a projective, maximal holonomy hyperkähler manifold with two non-collinear isotropic rational classes in $H^{1,1}(M)$ and with round Kähler cone. Assume that M satisfies the SYZ conjecture, that is, any nef bundle on M is semiample. Then M admits infinitely many transversal holomorphic Lagrangian fibrations. In particular, the Kobayashi pseudometric on M vanishes.

Proof: Since the Kähler cone of M is round, there exist rational vectors on the boundary of the Kähler cone of M. These points correspond to rational points in the real quadric $\{l \in \mathbb{P}H^2(M, \mathbb{Q}) \ \mid\ q(l, l) = 0\}$.

Each of such points corresponds to a Lagrangian fibration, because we assume that the SYZ conjecture holds. ■

Theorem 2.8: [KLV] Let (M, I) be a maximal holonomy, compact hyperkähler manifold with non-maximal Picard rank. Suppose that it has a deformation which has two transversal Lagrangian fibrations. Then the Kobayashi pseudometric on (M, I) vanishes.

Proof: The vanishing of the Kobayashi pseudometric then immediately follows from Theorem 2.7 and Theorem 2.4. Indeed, there exists a complex structure I' with vanishing Kobayashi pseudometric, and a sequence of diffeomorphisms such that $\lim_i \nu_i(I) = I'$. Then the Kobayashi pseudometric of (M, I) vanishes by semicontinuity properties of the diameter of the Kobayashi pseudometric. ■

Theorem 2.9: Let M be a compact, maximal holonomy hyperkähler manifold with $b_2(M) \geqslant 6$. Suppose that all its deformations satisfy the SYZ conjecture. Then the Kobayashi pseudometric on M vanishes.

Proof: See Theorem 2.7, Theorem 2.4 and Theorem 2.5. ■

Remark 2.10: All known examples of hyperkähler manifolds have $b_2(M) \geqslant 7$ and satisfy the SYZ conjecture. By the results above, the Kobayashi pseudometric of all known manifolds vanishes, unless their Picard rank is maximal.

Remark 2.11: The SYZ conjecture is true for all known hyperkähler examples, i.e., for deformations of Hilbert schemes of points on K3 surfaces (Bayer-Macrì [BM]; Markman [Mar3]), for deformations of the generalized Kummer varieties (Yoshioka [Y]), for O'Grady's sixfolds (Mongardi-Rapagnetta [MR]), and for O'Grady's tenfolds (Mongardi-Onorati, [MO]).

3 Main results

Here is a general result about lattices found in [KS, Satz 30.2], known as Kneser's orbit theorem.

Theorem 3.1: Let Λ be a non-degenerate integer lattice, and $O(\Lambda)$ its isometry group. Then for any $D > 0$, the group $O(\Lambda)$ acts with finitely many orbits on the set of non-degenerate sublattices $\Lambda' \subset \Lambda$ with discriminant $\mathsf{disc}(\Lambda') \leqslant D$.

The main technical theorem of this note is the following.

Theorem 3.2: Fix a natural number A. Let (Λ, q) be a non-degenerate indefinite lattice containing an isotropic vector. Then Λ contains a primitive sublattice Λ_1 isomorphic to $U(N) = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$ for some integer $N > A$.

Proof. Step 1: Let Λ be a lattice which does not contain primitive hyperbolic lattices with $U(N) \subset \Lambda$ with $N > A$. By Theorem 3.1 it follows that $O(\Lambda)$ acts with finitely many orbits on the set of all hyperbolic lattices in Λ. We will prove that this is impossible.

Step 2: Let S_1 be the set of primitive hyperbolic lattices in a hyperbolic lattice Λ, and S_2 the set of pairs (x, y) of non-orthogonal primitive isotropic vectors up to a permutation and a sign change. The group $O(U(N))$ of isometries of $U(N)$ is $(\mathbb{Z}/2\mathbb{Z})^2$. Indeed, a primitive isotropic vector can be mapped only to another primitive isotropic vector, and there are only 4 of them. From this observation it follows that the natural map $S_1 \longrightarrow S_2$ mapping a lattice to its primitive isotropic vectors is bijective. Therefore, the set of equivalence classes $\frac{S_1}{O(\Lambda)}$ is finite if and only if $\frac{S_2}{O(\Lambda)}$ is finite.

Step 3: Let $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $Q \subset \mathbb{P}V$ be the quadric defined by the equation $q = 0$. The primitive isotropic vectors in Λ are determined uniquely up to a sign by the rational points in Q. To prove that $O(\Lambda)$ does not act with finitely many orbits on the set of pairs (x, y) of non-orthogonal primitive isotropic vectors, it would suffice to show that $O(\Lambda)$ does not act with finitely many orbits on the set of pairs $(u, v) \in Q \times Q$ of rational, non-orthogonal points.

Step 4: In this step we prove that $O(\Lambda)$ does not act with finitely many orbits on the set of pairs $(u, v) \in Q \times Q$, where $u \in Q$ is fixed. In this case, the set of such v is in bijective correspondence with the rational lines passing through u, and not orthogonal to u (indeed, any rational line transversally intersecting quadric intersects it twice). Therefore, it would suffice to show that the stabilizer $\mathrm{St}_u(O(\Lambda))$ does not act with finitely many orbits on the set of rational projective lines $l \subset \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$ passing through u and not orthogonal to u.

Step 5: Let $\Lambda_1 := \frac{u^\perp}{u}$ be the quotient of the lattice $u^\perp \subset \Lambda$ by $\langle u \rangle$. Then $\mathrm{St}_u(O(\Lambda))$ acts on Λ_1, and each line $l \subset \mathbb{P}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$ produces a rank 1 rational subspace in $\Lambda_1 \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural action of $\mathrm{St}_u(O(\Lambda))$ factorizes through $O(W)$.

To prove that $\mathrm{St}_u(O(\Lambda))$ acts with infinitely many orbits on the set of projective lines passing through u, it would suffice to prove that $O(W)$ acts with infinitely many orbits on $\mathbb{P}(W \otimes_{\mathbb{Z}} \mathbb{Q})$. This is clear, because each 1-dimensional subspace in W has an integer invariant - the square of the smallest primitive vector - and this square can be arbitrary large. ■

Here is the main application of Theorem 3.2.

Theorem 3.3: Let M be a compact maximal holonomy hyperkähler manifold with $b_2(M) \geqslant 4$, satisfying the SYZ conjecture. Assume that $H^2(M, \mathbb{Q})$ has non-zero isotropic vectors. Then M admits a deformation with two distinct Lagrangian fibrations. If, in addition, M satisfies one of the two assumptions

(a) $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) = 0$,

(b) $b_2(M) \geqslant 6$, and M has Picard lattice of non-maximal rank,

then the Kobayashi pseudometric on M vanishes.

Proof: Consider a primitive lattice $\Lambda \subset H^2(M, \mathbb{Z})$ of signature (p, q), $p \leqslant 1, q \leqslant b_2 - 3$. Using the global Torelli theorem [V1], we can find a deformation (M, I_1) of M with Picard lattice Λ.

Since the rank of the indefinite lattice $H^2(M, \mathbb{Z})$ is at least 5, by Meyer's theorem [Me] there exists an isotropic vector $x \in H^2(M, \mathbb{Z})$. Applying Theorem 3.2, we can find a primitive hyperbolic sublattice $U(N) \subset H^2(M, \mathbb{Z})$ with $N > |q(v, v)|$ for all integer $(1, 1)$-classes. Choose the complex structure I such that $U(N)$ is the Picard lattice $H^{1,1}_I(M, \mathbb{Z})$ of (M, I). Then (M, I) has round Kähler cone, hence both integer isotropic generators of $U(N)$ are nef.

These classes correspond to Lagrangian fibrations since (M, I) satisfies SYZ, hence Kobayashi metric of (M, I) vanishes. This takes care of the first statement of Theorem 3.3.

Applying Theorem 2.4, we obtain that the Kobayashi metric vanishes for all ergodic complex structures, that is, for all complex structures I_1 such that $(H^{2,0}(M) \oplus H^{0,2}(M)) \cap H^2(M, \mathbb{Q}) = 0$. This proves the case (a) of Theorem 3.3.

It remains to prove Theorem 3.3 (b). Suppose that $(H^{2,0}(M, I) \oplus H^{0,2}(M, I)) \cap H^2(M, \mathbb{Q})$ has rank one and is generated by α, Since $\alpha^\perp$ has rank $\geqslant 5$, it contains isotropic vectors. Applying Theorem 3.2 again, we find a deformation (M, I') of M which satisfies $H^{1,1}_{I'}(M, \mathbb{Z}) = U(N)$ and $(H^{2,0}(M, I') \oplus H^{0,2}(M, I')) \cap H^2(M, \mathbb{Q}) = \langle \alpha \rangle$. For an appropriate choice of diffeomorphisms $\pi_i \in \operatorname{Diff}(M)$, the sequence $\nu_i(I')$ converges to I (by Theorem 2.3), hence the Kobayashi metric on (M, I) also vanishes. ■

Corollary 3.4: All known compact hyperkähler examples have vanishing Kobayashi pseudometric.

Proof: See Remark 2.11 and Theorem 3.3. ■

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